APPENDIX II

lerch's theorem

Let $f(t)$ be defined on $0 \le t < \infty$, and be piecewise continuous on every finite interval $0 \le t \le A$. Assume, moreover, that $f(t)$ is of exponential order, i.e., that there exist constants α and M such that $|f(t)| \le Me^{\alpha t}$, $0 \le t < \infty$. It is the purpose of this appendix to demonstrate the following theorem.

Theorem. (Lerch's theorem.) *If* $\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)\,dt$ *is identically zero for all* $s > s_0$, s_0 *some constant, then* $f(t)$ *is identically zero (except possibly at its points of discontinuity).*

Proof. Let $\phi(s) = \int_0^\infty e^{-st} f(t)\,dt$, $s > s_0$. Then if

$$P(x) = \sum_{k=0}^{n} a_k x^k$$

is any polynomial with real coefficients, we have

$$\begin{aligned}\int_0^\infty e^{-st} P(e^{-t}) f(t)\,dt &= \int_0^\infty e^{-st} \sum_{k=0}^{n} a_k e^{-kt} f(t)\,dt \\ &= \sum_{k=0}^{n} a_k \int_0^\infty e^{-st}[e^{-kt} f(t)]\,dt \\ &= \sum_{k=0}^{n} a_k \phi(s+k) = 0, \quad s > s_0.\end{aligned}$$

Making the change of variable $x = e^{-t}$, this last condition transforms to

$$\int_0^1 x^{s-1} P(x) f(-\ln x)\,dx = 0, \quad s > s_0.$$

Now choose a fixed $s_1 > \max\{s_0, 1, \alpha + 1\}$. Then

$$x^{s_1-1}|f(-\ln x)| \le Mx^{s_1-1} e^{\alpha(-\ln x)} = Mx^{s_1-(\alpha+1)},$$

and it follows that the function

$$G(x) = x^{s_1-1} f(-\ln x), \quad 0 < x \le 1,$$

tends to zero as $x \to 0$. Let us define $G(0) = 0$, thus making G continuous at $x = 0$. Then G is a function which is bounded in the interval $0 \le x \le 1$, has only "jump" discontinuities in this interval (although there may be infinitely many such discontinuities), and satisfies

$$\int_0^1 G(x)P(x)\,dx = 0$$

for every polynomial P. We shall deduce from these conditions that $G(x) = 0$ for $0 \le x \le 1$ (except possibly at its points of discontinuity). In fact, let us choose a complete orthogonal basis for the vector space $\mathcal{PC}[0, 1]$ with inner product $\mathbf{f} \cdot \mathbf{g} = \int_0^1 f(x)g(x)\,dx$.* Then any piecewise continuous function g which satisfies

$$\int_0^1 g(x)P(x) = 0$$

for every polynomial must be identically zero (except where it is discontinuous), for it is orthogonal to every member of the chosen basis and hence must be the zero vector in $\mathcal{PC}[0, 1]$.

Except for the fact that G may have infinitely many discontinuities, the proof would be complete. Fortunately, it is not difficult to show that a complete orthonormal basis for $\mathcal{PC}[0, 1]$ is also a complete basis for the slightly larger class of functions which, like G, may have infinitely many jump discontinuities but are bounded.† We thus conclude that G is identically zero wherever it is continuous, and hence since $G(x) = x^{s_1-1} f(-\ln x)$, the same must be true of f. ∎

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* We may choose, for example, the even-numbered Legendre polynomials as such a basis. For if

$$\int_0^1 g(x)P_{2k}(x)\,dx = 0, \quad k = 0, 1, 2, \ldots,$$

then for the *even extension* $\bar{g}$ of g to $-1 \le x \le 1$, we have

$$\int_{-1}^1 \bar{g}(x)P_n(x)\,dx = 0, \quad n = 0, 1, 2, \ldots$$

Thus the piecewise continuous function $\bar{g}$ must be zero (except at its points of discontinuity) and hence the same is true of g.

† The function $G(x)$ is, in fact, piecewise continuous in every interval $A \le x \le 1$ (for $0 < A < 1$) and is continuous at $x = 0$.