```
In[1]:= DSolve[x^2 (1 - x^2) y''[x] + x (4 x^2 - 5) y'[x] + (8 - 6 x^2) y[x] == 0, y[x], x]
Out[1]= {{y[x] → x^2 C[1] - x^2 √(1 - x^2) C[2]}}
```

Hledaným monomem je $r(x) = x^2$ ✓ ⇒ substituujeme: $y(x) = x^2 \cdot w(x)$ ✓

$\Rightarrow y' = 2xw + x^2 w'$ & $y'' = x^2 w'' + 4x w' + 2w$

Dosazení:

$$(x^2 - x^4)(x^2 w'' + 4x w' + 2w) + (4x^3 - 5x)(2xw + x^2 w') + (8 - 6x^2) x^2 w = 0$$

$$x^4 (1 - x^2) \cdot w'' + [4x^3 - 4x^5 + 4x^5 - 5x^3] w' + [2x^2 - 2x^4 + 8x^4 - 10x^2 + 8x^2 - 6x^4] w = 0$$

$$x^4 (1 - x^2) w'' - x^3 w' = 0 \quad \checkmark\checkmark$$

$$w'' - \frac{1}{x(1-x^2)} w' = 0 \quad \checkmark$$

Výpočet integračního faktoru: ✓

$$-\int \frac{1}{x(1-x^2)}\,dx = \int \left(\frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x} \right) dx = \left\| \begin{array}{l} A = -1 \\ B = -1/2 \\ C = +1/2 \end{array} \right\| =$$

$$= -\ln x + \frac{1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x) = -\ln \frac{\sqrt{1-x^2}}{x}$$

$$w'' - \frac{1}{x(1-x^2)} w' = 0 \quad / \; \frac{\sqrt{1-x^2}}{x} \quad \checkmark\checkmark \leftarrow \text{I.F.}$$

$$\left(w' \cdot \frac{\sqrt{1-x^2}}{x} \right)' = c'$$

$$w'(x) = \frac{cx}{\sqrt{1-x^2}} \quad \checkmark$$

$$\Rightarrow w(x) = \int \frac{cx}{\sqrt{1-x^2}}\,dx = \tilde{c}\sqrt{1-x^2} + D \quad \checkmark$$

$$\Rightarrow y(x) = Dx^2 + \tilde{c}x^2\sqrt{1-x^2} \quad \checkmark$$

$$Dom(y) = (-1, 1) \quad \checkmark$$

Nalezněte obecné formální řešení rovnice

$$y'(4x+17y-50)+x+4y-12=0.$$

Co je lépe formálním řešením? Které z formálních řešení procházejí bodem $(x_0,y_0)=(6,2)$?

Řešení: $f(x,y)+g(x,y)\cdot y'=0 \Rightarrow g(x,y)=4x+17y-50$, $f(x,y)=x+4y-12$ ✓

$\frac{\partial g}{\partial x}=\frac{\partial f}{\partial y}=4$ ⇒ jedná se o exaktní rovnici

$$H(x,y)=\int_0^x f(s,y)\,ds+\int_0^y g(0,\tau)\,d\tau \checkmark = \int_0^x (s+4y-12)\,ds+\int_0^y (17\tau-50)\,d\tau =$$

$$=\left[\frac{s^2}{2}+4ys-12s\right]_0^x+\left[\frac{17}{2}\tau^2-50\tau\right]_0^y=\frac{x^2}{2}+4xy-12x+\frac{17}{2}y^2-50y$$

$$x^2+17y^2+8xy-24x-100y=C \checkmark\checkmark$$

$$(x,y)\begin{pmatrix}1&4\\4&17\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}-2(x,y)\begin{pmatrix}12\\50\end{pmatrix}=C$$ ← ve standardizovaném tvaru pro kvadriky

- dělicí směr: $A\cdot\vec{s}=\vec{k} \Leftrightarrow \begin{pmatrix}1&4\\4&17\end{pmatrix}\begin{pmatrix}s_1\\s_2\end{pmatrix}=\begin{pmatrix}12\\50\end{pmatrix}$

$$\Rightarrow \vec{s}=\begin{pmatrix}4\\2\end{pmatrix} \checkmark\checkmark$$

- kalibrace konstanty C na základě požadavku, aby $(x_0,y_0)\in F$

$$36+68+96-144-200 \overset{!}{=} C \Rightarrow \underline{C=-144} \checkmark$$ (jediné řešení prochází bodem $(6,2)$!)

- úprava rovnice: $x^2+17y^2+8xy-24x-100y=-144$ $\quad -100y$

$$(x+4y-12)^2+y^2-144+96y=-144$$

$$\underline{(x+4y-12)^2+(y-2)^2=4}$$ ✓ ✓

Lagrangeův tvar ⇒ jedná se o elipsu ✓

Pro $x = 1$ máme číselnou řadu $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{1+n^2}}$, která diverguje, neboť

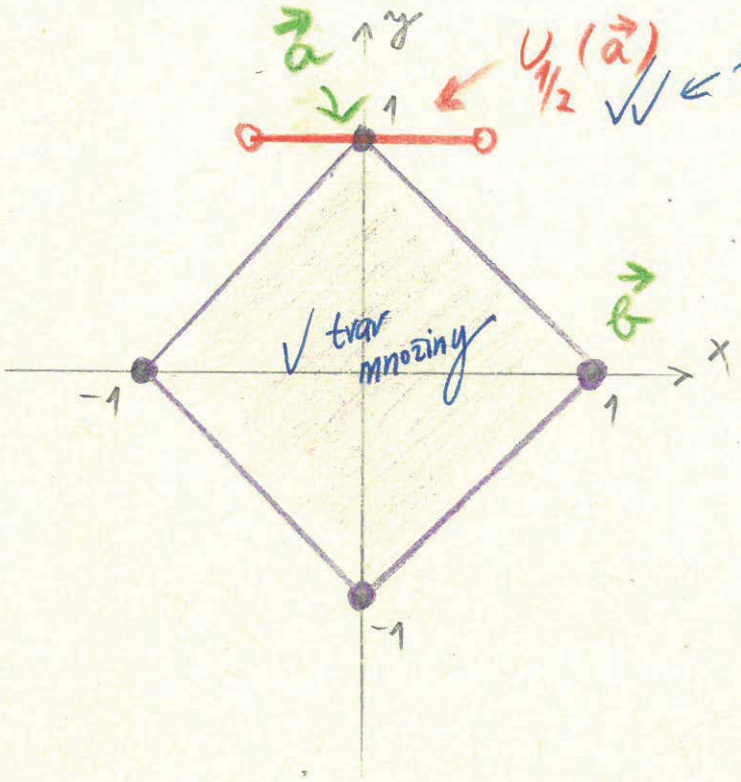
$$\frac{\ln n}{\sqrt{2}\cdot n} = \frac{\ln n}{\sqrt{n^2+n^2}} < \frac{\ln n}{\sqrt{1+n^2}}$$

$\Downarrow\Downarrow$

Minorantní řada $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{2}\cdot n}$ diverguje z integrálního kritéria, neboť

$$\int_1^{+\infty} \frac{\ln t}{t}\,dt = \Big| \begin{matrix} u = \ln t \\ u' = \frac{1}{t} \end{matrix} \Rightarrow du = \frac{1}{t}\,dt \Big| = \int_0^{\infty} u\,du = +\infty$$

$\Rightarrow$ zadaná funkční řada nekonverguje na $\mathbb{R}^+$ ani bodově ✓



✓✓ ← tvar okolí

$$\rho(\vec{x},\vec{z}) = |x_1 - z_1| + |x_2 - z_2|$$

$$U_{1/2}(0,0) = \{(x,y) \in \mathbb{R}^2 : \begin{matrix} y = 0 \\ x \in (-\frac{1}{2}; \frac{1}{2}) \end{matrix}\}$$

jde o úsečku délky 1 rovnoběžnou s osou x

$\vec{a}$ je izolovaný' $\Leftarrow$ $U_{1/2}(\vec{a}) \cap X = \{\vec{a}\}$ ✓✓

$\vec{b}$ je hromadný' $\Leftarrow$ $\forall \varepsilon > 0$: $U_\varepsilon(\vec{a}) \cap X \neq \{\vec{a}\}$ ✓✓

$$y^{(n)}(0) = \frac{3^{n+2}}{n+2}$$

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{3^{n+2}}{n!(n+2)} x^n = \frac{1}{x^2} \sum_{n=0}^{\infty} \frac{3^{n+2}}{n!(n+2)} x^{n+2} =: \frac{1}{x^2} s(x)$$

$$s'(x) = \sum_{n=0}^{\infty} \frac{3^{n+2}}{n!} x^{n+1} = 9x \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = 9x e^{3x}$$

$$s(x) = \left| \begin{matrix} u = 9x & v' = e^{3x} \\ u' = 9 & v = \frac{1}{3} e^{3x} \end{matrix} \right| = 3x e^{3x} - 3 \int e^{3x} dx + C = 3x e^{3x} - e^{3x} + C$$

$$s(0) \overset{!}{=} 0 \quad \Rightarrow \quad 0 - 1 + C = 0 \quad \Rightarrow \quad C = 1$$

$$s(x) = 3x e^{3x} - e^{3x} + 1$$

$$y(x) = \frac{3x e^{3x} - e^{3x} + 1}{x^2}$$ (no ale to má problém v bodě $x = 0$!)

Ale: $$\lim_{x \to 0} \frac{3x e^{3x} - e^{3x} + 1}{x^2} \overset{l'H}{=} \lim_{x \to 0} \frac{3e^{3x} + 9x e^{3x} - 3e^{3x}}{2x} = \frac{9}{2} = \frac{3^2}{2}$$

Hledanou analytickou funkcí je

$$y(x) = \begin{cases} \dfrac{3x e^{3x} - e^{3x} + 1}{x^2} & x \neq 0 \\ 9/2 & x = 0 \end{cases}$$

$x^2 - 2xy + u^2 - 2uv + 2v^2 - 2z = 2$ $\quad Q(x,y,z,u,v)$ v $\mathbb{R}^5$

$(x-y)^2 - y^2 + (u-v)^2 + v^2 - 2(z+1) = 0$ ✓✓

$\underbrace{\xi^2 - \eta^2 + \lambda^2 + \mu^2 - 2\varkappa = 0}_{\text{normální tvar v } \mathbb{R}^5}$ ✓ ← musí být naplněna definice norm. tvaru kvadriky!

$\Rightarrow \quad sg(Q) = (3,1,1)$ ✓

$sG(Q) = (4,2,0)$ ✓ $\Leftarrow \quad \xi^2 - \eta^2 + \lambda^2 + \mu^2 - 2\varkappa\omega = 0$

$\varkappa = \tilde{\varkappa} + \tilde{\omega}$

$\omega = \tilde{\varkappa} - \tilde{\omega}$

$\varkappa\omega = \tilde{\varkappa}^2 - \tilde{\omega}^2$

kvadrika je regulární v $\mathbb{R}^5$! ✓

$$\left.\begin{aligned} \xi &= x - y \\ \eta &= y \\ \lambda &= u - v \\ \mu &= v \\ \varkappa &= z + 1 \end{aligned}\right\} \Rightarrow \begin{aligned} x &= \xi + \eta \\ y &= \eta \\ u &= \lambda + \mu \\ v &= \mu \\ z &= \varkappa - 1 \end{aligned}$$

$$\left.\begin{pmatrix} x \\ y \\ u \\ v \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \lambda \\ \mu \\ \varkappa \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}\right\}$$ ✓✓

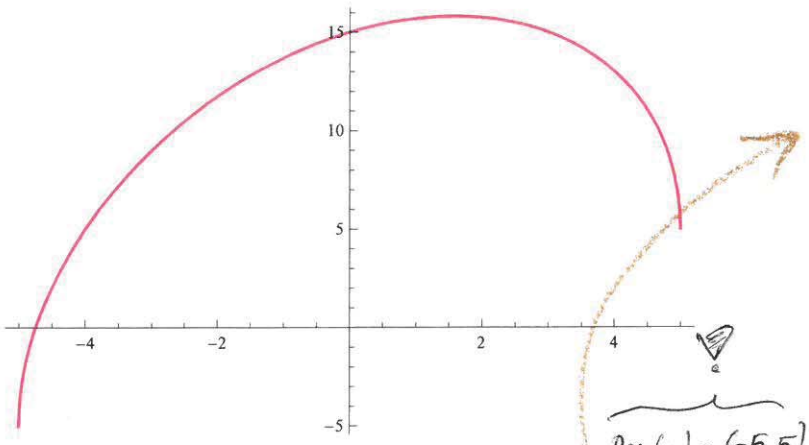
$(25-x^2)y'' - xy' + y = 0 \qquad (x_0, y_0) = (4, 13)$

$L(y(x)) = 0$ řeší funkce $z(x) = x$

$\Rightarrow y = x \cdot z$ & $y' = z + xz'$ & $y'' = 2z' + xz''$ ✓ ✓

In[2]:=
```
Plot[x + 3 Sqrt[25 - x^2], {x, -5, 5}, PlotStyle →
 Directive[RGBColor[1., 0., 0.61], Opacity[1.], AbsoluteThickness[1.71]]]
```

Out[2]=



$y(4) = 13$

$13 = 4D + 3E$

$y(\sqrt{\tfrac{5}{2}}) = 0$

$y'(x) = D - E\dfrac{x}{\sqrt{25-x^2}}$

$y'(\sqrt{\tfrac{5}{2}}) = D - \dfrac{E}{3} = 0$

$\Rightarrow (D, E) = (1, 3)$

$Dom(y) = (-5, 5]$ ✓

$y(x) = x + 3\sqrt{25-x^2}$ ✓✓

In[3]:=
```
FullSimplify[
 (25 - x^2) * D[x + 3 Sqrt[25 - x^2], {x, 2}] - x * D[x + 3 Sqrt[25 - x^2], {x, 1}] + x + 3 Sqrt[25 - x^2]]
```

Out[3]= 0

In[4]:= `x = 4`

Out[4]= 4

In[5]:= `x + 3 Sqrt[25 - x^2]`

Out[5]= 13

$(25-x^2)(2z' + xz'') - xz - x^2z' + xz = 0$

$x \cdot (25-x^2)z'' - z'(+2x^2 - 30 + x^2) = 0$

$x(25-x^2)w' + w(50 - 3x^2) = 0$

$w' + \dfrac{50-3x^2}{x(5-x)(5+x)}w = 0$ ✓✓

Parciální zlomky: $\dfrac{50-3x^2}{x(5-x)(5+x)} = \dfrac{A}{x} + \dfrac{B}{5-x} + \dfrac{C}{5+x} \Rightarrow \begin{matrix} A = 2 \\ B = -1/2 \\ C = 1/2 \end{matrix}$ ✓ ✓

Integrační faktor:

$\exp\left[\int \frac{2}{x}dx - \frac{1}{2}\int \frac{1}{5-x}dx + \frac{1}{2}\int \frac{1}{x+5}dx\right] = \exp\left[\ln x^2 + \ln\sqrt{5-x} + \ln\sqrt{x+5}\right] =$

$= x^2\sqrt{25-x^2}$ ✓

$\left(x^2 \cdot \sqrt{25-x^2} \cdot w(x)\right)' = C'$

$w(x) = \dfrac{C}{x^2\sqrt{25-x^2}}$ ✓

$z(x) = C\int \frac{1}{x^2\sqrt{25-x^2}}dx = \left| \begin{matrix} x = 5\sin\tau \\ dx = 5\cos\tau\, d\tau \end{matrix} \right| = \int \frac{5\cos\tau \cdot C}{25\sin^2\tau \cdot 5\cos\tau}d\tau = -\frac{C}{25}\operatorname{ctg}(\tau) =$

$= -\frac{C}{25}\sqrt{\frac{\cos^2(\tau)}{\sin^2(\tau)}} = -\frac{C}{25}\sqrt{\frac{25-x^2}{x^2}} = -\frac{C\sqrt{25-x^2}}{25x} + D \Rightarrow y(x) = Dx + E\sqrt{25-x^2}$ ✓ ✓

$\langle f|g\rangle = 35 \int_0^1 f(x)\, g(x)\, x^2\, dx$ ✓

$x^2 \in U_{1/2}(x) \Leftrightarrow \rho(x, x^2) < \frac{1}{2}$ ✓

$x^2 \in U_{1/2}(x)$ ✓

$\rho^2(x, x^2) = \| x - x^2 \|^2 = \langle x - x^2 \mid x - x^2 \rangle = 35 \int_0^1 (x - x^2)^2 x^2\, dx =$ ✓

$= 35 \int_0^1 (x^4 - 2x^5 + x^6)\, dx = 35 \left(\frac{1}{5} - \frac{2}{6} + \frac{1}{7} \right) =$

$= 35 \left(\frac{1}{5} - \frac{1}{3} + \frac{1}{7} \right) = 35 \cdot \frac{21 - 35 + 15}{35 \cdot 3} = \frac{1}{3}$ ✓

$\rho(x, x^2) = \frac{1}{\sqrt{3}} \times \frac{1}{2}$ ✓

$\Rightarrow x^2 \notin U_{1/2}(x)$ ✓

neto

- $q(1,3,-1) = -2-9-20-12+14+30 = -43+44 = 1$ ✓

- $\vec{u} = (x,y,z) = ?$

$\langle (0,0,1)^T | \vec{u} \rangle \overset{!}{=} 0 \Rightarrow z = 0$ (interpretace zadání $\langle (0,0,1)^T | \vec{u} \rangle = 0$ je jasná! Musí jít o standardní skalární součin, protože $q(\vec{u}) \neq 0$!!) ✓

$$(x,y,z)\begin{pmatrix} -2 & -2 & -7 \\ -2 & -1 & -5 \\ -7 & -5 & -20 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = -x - 2z \overset{!}{=} 0 \Rightarrow x = -2z \;\&\; z = 0$$

$\Rightarrow x = 0$

to je vidět hned, neboť $-2 < 0$!! (viz Sylvestr. kritérium)

$q(0,y,0) = -y^2 = 0, -1, 1$

$\Rightarrow y = 1$

$\Rightarrow \vec{u} = (0,1,0)$ ✓

- $\vec{v} = (x,y,z) = ?$

$$(x,y,z)\begin{pmatrix} -2 & -2 & -7 \\ -2 & -1 & -5 \\ -7 & -5 & -20 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = -x - 2z \overset{!}{=} 0$$

$$(x,y,z)\begin{pmatrix} -2 & -2 & -7 \\ -2 & -1 & -5 \\ -7 & -5 & -20 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -2x - y - 5z \overset{!}{=} 0$$

$\Rightarrow z = \alpha,\ x = -2\alpha,\ y = 4\alpha - 5\alpha = -\alpha$

$q(-2\alpha, -\alpha, \alpha) = \alpha^2 q(-2,-1,1) = \alpha^2(-8-1-20-8+28+10) = \alpha^2 = 0, -1, 1$

$\Rightarrow \alpha = 1 \Rightarrow \vec{v} = (-2,-1,1)$ ✓

- $B = \left\{ \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right\}$ ✓

- $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ ✓

- normální tvar: $q(a,b,c) = a^2 - b^2 + c^2$ ✓

$$\sum_{n=2}^{\infty} \underbrace{\frac{(-1)^n}{\ln(n)}}_{f_n(x)} \underbrace{\frac{n}{\sqrt{8n^2+x^2}}}_{g_n(x)} \rightarrow$$ členění

a) společný předpoklad o monotónii $g_n(x)$:

$$g_n^2(x) < g_{n+1}^2(x)$$

$$\frac{n^2}{8n^2+x^2} < \frac{(n+1)^2}{8(n+1)^2+x^2}$$

$$8n^2(n+1)^2 + n^2x^2 < 8n^2(n+1)^2 + x^2$$

$$n^2 < (n+1)^2$$

b) Abelovo I. $\sum_n \frac{(-1)^n}{h(n)}$ stejnoměrně konverguje na $\mathbb{R}$, neboť jde o řadu konst. funkcí, kde stačí Leibnizovým kritériem vyšetřit bodovou konvergenci:

$$\lim_{n\to\infty} \frac{1}{h(n)} = 0. \quad \wedge \quad \frac{1}{\ln(n)} > \frac{1}{\ln(n+1)} \quad \wedge \text{ střídá znaménka}$$

c) Abelovo II omezenost $g_n(x)$:

$$|g_n(x)| = \left| \frac{n}{\sqrt{8n^2+x^2}} \right| \leq \frac{1}{\sqrt{8}}$$

d) Dirichletovo I. posloup. část. součtu řady $\sum f_n(x)$ je jistě omezená, neboť již bylo prokázáno, že $\sum_n f_n(x)$ SK na $\mathbb{R}$

e) Dirichletovo II. $g_n(x) \overset{?}{\rightrightarrows} 0$

$$\lim_{n\to\infty} \frac{n}{\sqrt{8n^2+x^2}} = \frac{1}{\sqrt{8}} \neq 0 \Rightarrow g_n(x) \not\to 0 \Rightarrow g_n(x) \not\rightrightarrows 0$$

$\Downarrow$

Šlo užít pouze Abelovo kritérium, Dirichletovo nikoli.

$\hat{L} = \sum_{m=0}^{n} p_m(x) \frac{d^m}{dx^m}$ $\quad p_n(x) = 1$! $\quad p_m(x) \in \mathbb{C}(I)$ pro $m \in \underline{n-1}$

7

$p_0(x) = 0 \Rightarrow \hat{L}(1) = 0 \Rightarrow 1 \in F_S$ ✓

$e^{2x} \in F_S$ & $e^{2x}, 1$ jsou LN

$n = 3 \Rightarrow$ $\dim(F_S) = 3$ ✓ $\Rightarrow$ protože 2 členy známe, zbývá určit poslední (označíme ho $y(x)$)

$y(x) = ?$

$W_{1, e^{2x}, y(x)} \overset{!}{=} 4e^{2x}(2x-1)$

$$W_{1, e^{2x}, y(x)} = \begin{vmatrix} 1 & e^{2x} & y \\ 0 & 2e^{2x} & y' \\ 0 & 4e^{2x} & y'' \end{vmatrix} = 2e^{2x}y'' - 4e^{2x}y' \overset{!}{=} 4e^{2x}(2x-1)$$ ✓

odtud:

$y'' - 2y' = 4x - 2$ ✓

$\lambda^2 - 2\lambda = \lambda(\lambda - 2) \overset{!}{=} 0 \Rightarrow \underbrace{\lambda = 0 \vee \lambda = 2}$

„0" je jednonásobným kořenem charakteristického polynomu

$\tilde{F}_S = \{1, e^{2x}\}$

$y_p(x) = (\alpha x + \beta) \cdot x^{(1)}$ ✓

$y_p'(x) = 2\alpha x + \beta$ & $y_p''(x) = 2\alpha$

Dosazení: $2\alpha - 4\alpha x - 2\beta \overset{!}{=} 4x - 2$

$\Rightarrow (\alpha, \beta) = (-1, 0)$

$y(x) = (C) + (D e^{2x}) - x^2$ ✓

OBA ČLENY VŽ ALE VE F_S JSOU $\Rightarrow$

$\Rightarrow F_S = \{1, e^{2x}, -x^2\}$ ✓ (lze i $F_S = \{1, e^{2x}, x^2\}$)

(to souvisí s pořadím funkcí ve Wronskiánu)

$$\int_0^{\pi/4} \frac{1-\cos x}{x^2}\,dx = \int_0^{\pi/4} \frac{1-\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}}{x^2}\,dx = \int_0^{\pi/4} \frac{1}{x^2}\sum_{k=1}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}\,dx =$$

$$= \int_0^{\pi/4} \sum_{k=1}^{\infty}(-1)^k \frac{x^{2k-2}}{(2k)!}\,dx = \left| \; k-1=m \; \right| = \int_0^{\pi/4} \sum_{m=0}^{\infty}(-1)^{m+1} \frac{x^m}{(2m+2)!}\,dx =$$

$$= \left| \begin{array}{c} \text{zdůvodnění záměny} \\ \text{(viz níže)} \end{array} \right| = \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(2m+2)!} \int_0^{\pi/4} x^m\,dx = \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(2m+2)!} \left[\frac{x^{m+1}}{m+1}\right]_0^{\pi/4} =$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(2m+2)!} \frac{1}{m+1} \left(\frac{\pi}{4}\right)^{m+1}$$

Řada $\sum_{m=0}^{\infty}(-1)^{m+1}\frac{x^m}{(2m+2)!}$ má obor konvergence rovný množině $\mathbb{R}$. Uzavřený interval $Y = \langle 0; \pi/4 \rangle$ je jeho podmnožinou. Podle základní věty teorie mocninných řad tedy $\sum (-1)^{m+1}\frac{x^m}{(2m+2)!} \overset{Y}{\equiv}$ $\Rightarrow$ záměna je možná podle věty o integrování řad člen po členu

11c.

$$y' = \frac{\frac{y^2}{x^2} - 2\frac{y}{x} - 1}{\frac{y^2}{x^2} + 2\frac{y}{x} - 1}$$ ... homogenní rovnice!!

– lineární rešení nás nezajímá (∈ formálním řešením není přímka)

$\Rightarrow$ $z = \frac{y}{x} \Rightarrow y = z \cdot x$ a $y' = z'x + z$

$$z'x + z = \frac{z^2 - 2z - 1}{z^2 + 2z - 1}$$

$$z'x = \frac{z^2 - 2z - 1 - z^3 - 2z^2 - z}{z^2 + 2z - 1}$$

$$\frac{z^2 + 2z + 1}{z^3 + z^2 + z + 1} z' = -\frac{1}{x}$$

$z^3 + z^2 + z + 1 = (z^2 + 1)(z + 1)$

$$\frac{z^2 + 2z + 1}{z^3 + z^2 + z + 1} = \frac{A}{z+1} + \frac{Bz + C}{z^2 + 1}$$

$$Az^2 + A + Bz^2 + Cz + Bz + C = z^2 + 2z - 1$$

$$\left.\begin{array}{l} A + B = 1 \\ B + C = 2 \\ A + C = -1 \end{array}\right\} \Rightarrow (A, B, C) = (-1, 2, 0)$$

$$\int \frac{-1}{z+1}\,dz + \int \frac{2z}{z^2+1}\,dz = -\int \frac{1}{x}\,dx$$

$$\ln \frac{z^2+1}{|z+1|} = \ln \frac{c}{|x|}$$

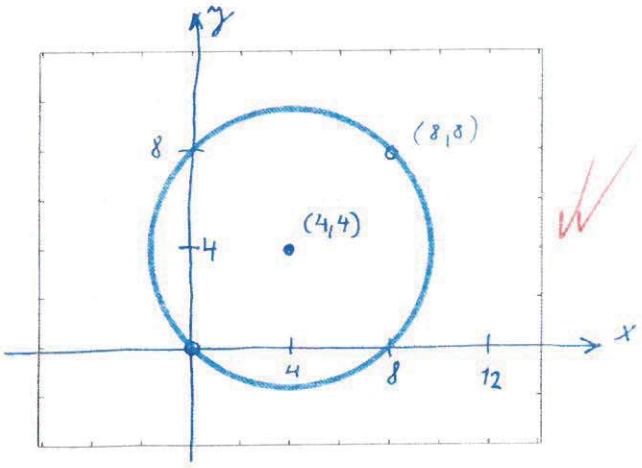
$$z^2 + 1 = \frac{c}{x}(z+1) \Rightarrow \frac{y^2}{x^2} + 1 = \frac{c}{x}\left(\frac{y}{x} + 1\right)$$

$$x^2 + y^2 = c(x + y)$$

$$\left(x - \frac{c}{2}\right)^2 + \left(y - \frac{c}{2}\right)^2 = \frac{c^2}{2}$$

kružnice se středem v bodě $\left(\frac{c}{2}; \frac{c}{2}\right)$
hledaný poloměr $\frac{c}{\sqrt{2}}$

$\Rightarrow$ $\frac{c}{\sqrt{2}} = 4\sqrt{2} \Rightarrow c = 8$ $\Rightarrow$ střed kružnice: $S = (4, 4)$



•) $q(\vec{x}) = \vec{x}^T A \vec{x} \Rightarrow q(\vec{x}+\vec{y}) = (\vec{x}+\vec{y})^T A (\vec{x}+\vec{y}) =$

$= \vec{x}^T A \vec{x} + \vec{x}^T A \vec{y} + \vec{y}^T A \vec{x} + \vec{y}^T A \vec{y} =$

$= q(\vec{x}) + 2qq(\vec{x},\vec{y}) + q(\vec{y})$ ✓

•) $q(\alpha\vec{x}) = (\alpha\vec{x})^T A (\alpha\vec{x}) = \alpha^2 \vec{x}^T A \vec{x} = \alpha^2 q(\vec{x})$ ✓

•) $B_p = \{\vec{u}, \vec{v}, \vec{w}\} \Rightarrow q(\vec{u},\vec{v}) = qq(\vec{u},\vec{w}) = qq(\vec{v},\vec{w}) = 0$

$\Delta_1 = -8 \wedge \Delta_2 = 4 \wedge \Delta_3 = -2 \Rightarrow$ Sylvesterovo kritérium ✓ $\Rightarrow q(\vec{x}) \neq 0 \Rightarrow$

$\Rightarrow q(\vec{u}) = q(\vec{v}) = q(\vec{w}) = -1$ ✓

Závěr:

$q(\vec{x}+2\vec{v}+4\vec{w}) = q(\vec{u}) + 2^2 q(\vec{v}) + 4^2 q(\vec{w}) \neq 0 + 0 + 0 =$

$= -1 - 4 \cdot 1 - 16 \cdot 1 = -21$

✓

Jednoduché řešení:

Rovnice $(1-x^2)y'' - xy' + y = 0$ zcela zjevně řeší funkce $y(x) = x$, která je LN s $\sqrt{x^2-1}$ ⇒ známe celý fundamentální systém ⇒ ✓

$$\Rightarrow \quad y(x) = \alpha x + \beta\sqrt{x^2-1} \quad \checkmark\checkmark$$

$$Dom(y) = (1; +\infty) \quad \checkmark$$

Méně napadité řešení:

- snížení řádu substitucí: $y(x) = z(x)\sqrt{x^2-1}$ ✓

$$y' = z'\sqrt{x^2-1} + z\frac{x}{\sqrt{x^2-1}}$$

$$y'' = z''\sqrt{x^2-1} + 2z'\frac{x}{\sqrt{x^2-1}} + z\frac{\sqrt{x^2-1} - \frac{x^2}{\sqrt{x^2-1}}}{x^2-1}$$

Pozor:

$$-z''(x^2-1)^{3/2} - 2z'x(x^2-1)^{1/2} - xz'(x^2-1)^{1/2} = 0$$

$$z'' + \frac{3x}{x^2-1}z' = 0 \quad \checkmark \quad / \; IF = e^{\frac{3}{2}\ln(x^2-1)} \equiv (x^2-1)^{3/2} \quad \checkmark$$

$$\left(z' \cdot (x^2-1)^{3/2}\right)' = 0$$

$$z' = \frac{C}{(x^2-1)^{3/2}} \quad \checkmark$$

to pro $x \in \mathbb{R}$ úplně nejde viz (*) na dobré straně

$$\Rightarrow z(x) = \int \frac{\tilde{C}}{(1-x^2)^{3/2}}\,dx = \left| \begin{matrix} x = \cos\varphi \\ y = -\sin\varphi\,d\varphi \end{matrix} \right| = \int \frac{\tilde{\tilde{C}}}{\sin^3(\varphi)}\sin\varphi\,d\varphi =$$

$$= \int \frac{\tilde{\tilde{C}}}{\sin^2(\varphi)}\,d\varphi = \alpha \cdot \operatorname{cotg}(\varphi) + \beta = \alpha\frac{x}{\sqrt{1-x^2}} + \beta$$

✓

$$y(x) = \tilde{\alpha}x + \beta\sqrt{x^2-1} \quad \checkmark$$

$$Dom(y) = (1; +\infty) \quad \checkmark$$

tady je 8 bodů, ale to je OK

$$dist(x, A) = \inf \{ \rho(x; g_a(x)) : g_a(x) \in A \}$$

$$\rho^2(x; g_a(x)) = \|x - g_a(x)\|^2 = \langle x - g_a(x) \mid x - g_a(x) \rangle = \int_0^1 (x - g_a(x))^2 dx =$$

$$= \int_0^1 4(a^2x^3 + 6 - 6ax^2)\, dx = \left[a^2x^4 + 24x - 8ax^3 \right]_0^1 = a^2 + 24 - 8a$$

Hledáme minimum funkce $\omega(a) := a^2 + 24 - 8a$

$$\frac{d\omega}{da}(a) = 2a - 8 \stackrel{!}{=} 0$$

$a = 4 \Rightarrow$ zcela zjevně jde o minimum

($\omega(a)$ je kvadratická parabola)

$$\Rightarrow dist(x, A) = \min \sqrt{a^2 + 24 - 8a} = \sqrt{\omega(4)} = \sqrt{16 + 24 - 32} = \sqrt{8} = 2\sqrt{2}$$

$$dist(x, A) = 2\sqrt{2}$$

10 b

$$\sum_{n=1}^{\infty} (-1)^n \frac{(4n)!!}{(4n+1)!!} \frac{\cosh(nx)+\sinh(nx)}{\cosh(nx)} \quad \text{SK na } \langle 0, +\infty)$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{(4n)!!}{(4n+1)!!} \frac{e^{nx}+e^{-nx}+e^{nx}-e^{-nx}}{e^{nx}+e^{-nx}} = 2\sum_{n=1}^{\infty} (-1)^n \frac{(4n)!!}{(4n+1)!!} \underbrace{\frac{1}{1+e^{-2nx}}}_{\text{úprava}}$$

W-kritérium nelze užít, neboť

$$\left| 2(-1)^n \frac{(4n)!!}{(4n+1)!!} \frac{1}{1+e^{-2nx}} \right| \leq 2 \frac{(4n)!!}{(4n+1)!!} \sup_{x\in\mathbb{R}^+} \left| \frac{1}{1+e^{-2nx}} \right| = \underbrace{2\frac{(4n)!!}{(4n+1)!!}}_{\text{majoranta}}$$

ale $\sum \frac{(4n)!!}{(4n+1)!!}$ je divergentní

$$\in \lim_{n\to\infty} n\left(1 - \frac{(4n+4)!!}{(4n+5)!!}\frac{(4n+1)!!}{(4n)!!}\right) = \lim_{n\to\infty} n\left(1 - \frac{(4n+4)(4n+2)}{(4n+5)(4n+3)}\right) =$$

$$= \lim_{n\to\infty} n \frac{32n+15-24n-8}{(4n+5)(4n+3)} = \frac{8}{16} = \frac{1}{2} \in (0;1)$$

– ale řada $\sum_{n=1}^{\infty} (-1)^n \frac{(4n)!!}{(4n+1)!!}$ tedy konverguje (alternativní Raabe)

Abelovo kritérium

$$f_n(x) = (-1)^n \frac{(4n)!!}{(4n+1)!!} \qquad g_n(x) = \frac{2}{1+e^{-2nx}}$$

a) $\sum f_n(x)$ konverguje stejnoměrně ($\in$ jde o číselnou posloupnost)

b) $|g_n(x)| = \left| \frac{2}{1+e^{-2nx}} \right| \leq 2$ omezenost

c) monotónie

$$\frac{1}{1+e^{-2nx}} < \frac{1}{1+e^{-(2n+2)x}}$$

$$e^{-2nx} e^{2x} < e^{-2nx}$$

$$e^{2x} < 1$$

SK na $\langle 0; +\infty)$

Pb

$$\sum_{n=1}^{\infty} \frac{\ln(n^x)}{\sqrt{x^4+n^2}} = \sum_{n=1}^{\infty} \frac{x\cdot\ln(n)}{\sqrt{x^4+n^2}} =: \sum_{n=1}^{\infty} h_n(x)$$ ✓

$$\lim_{n\to\infty} h_n(x) \overset{L'H}{=} \lim_{n\to\infty} x \cdot \frac{1/n}{\frac{n}{\sqrt{x^4+n^2}}} = x \cdot \lim_{n\to\infty} \frac{\sqrt{x^4+n^2}}{n^2} = 0$$

je tedy třeba zodpovědět otázku, zda $\frac{x\cdot\ln(n)}{\sqrt{x^4+n^2}} \overset{\mathbb{R}^+}{\rightrightarrows} 0$ ✓ (to dokazujeme)

Užijeme supremální kritérium:

$$\sigma_n := \sup_{x\in\mathbb{R}^+} |h_n(x)| = ?$$

$$h_n'(x) = \ln(n)\ \frac{\sqrt{x^4+n^2} - x\cdot\frac{1}{2}\frac{4x^3}{\sqrt{x^4+n^2}}}{x^4+n^2} = \ln(n)\ \frac{x^4+n^2-2x^4}{(x^4+n^2)^{3/2}} \overset{!}{=} 0$$ ✓

$$x^4 = n^2$$

$$x = \sqrt{n}$$ ✓

$\lim_{x\to 0} h_n(x) = 0 \ \wedge\ \lim_{x\to\infty} h_n(x) = 0 \ \wedge\ h_n(x) \in \mathcal{C}(\mathbb{R}^+)$ ✓ (nebo obrázek)

$\Rightarrow$ $x = \sqrt{n}$ je bodem (argumentem) maxima

$$\Rightarrow \sigma_n = h_n(\sqrt{n}) = \ln(n)\cdot\frac{\sqrt{n}}{\sqrt{n^2+n^2}} = \frac{\ln n}{\sqrt{n}}$$ ✓

$$\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}} \overset{L'H}{=} \lim_{n\to\infty} \frac{1/n}{\frac{1}{2}\frac{1}{\sqrt{n}}} =$$

$$= \lim_{n\to\infty} 2\cdot\frac{1}{\sqrt{n}} = 0$$

$\Rightarrow$ nutná podmínka je splněna ✓

9b

In[7]:= **Expand[(x - 2 y - 7 z)^2 + y z]**

Out[7]= $x^2 - 4xy + 4y^2 - 14xz + 29yz + 49z^2$ $= (x-2y-7z)^2 + yz$

a; $b+c$; $b-c$ ← trik, jak pracovat se součinem yz

$= q(a,b,c) = \underbrace{a^2 + b^2 - c^2}_{\text{normální tvar}} = 0$ ⇒ $\nabla q(Q) = (2,1,0)$ ✗

$\mathcal{H}_q(Q) = (2,1,1)$ ✗

singulární

složená $\vec{s} = \vec{0}$

$$\left.\begin{aligned} a &= x-2y-7z \\ y &= b-c \\ z &= b+c \end{aligned}\right\} \Rightarrow \begin{aligned} a &= x-2y-7z \\ b &= \tfrac{1}{2}(y+z) \\ c &= \tfrac{1}{2}(z-y) \end{aligned}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & -2 & -7 \\ 0 & \tfrac{1}{2} & \tfrac{1}{2} \\ 0 & -\tfrac{1}{2} & \tfrac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Obráceně:

$$x = a + 2(b-c) + 7(b+c) = a + 9b + 5c$$
$$y = b - c$$
$$z = b + c$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left(\begin{array}{ccc|c} 1 & 9 & 5 & a \\ 0 & 1 & -1 & b \\ 0 & 1 & 1 & c \end{array}\right)$$

$$B_p = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 9 \\ 1 \\ 1 \end{pmatrix}; \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$g(x) = \frac{1}{\sqrt{x-3}} \qquad x_0 = 7 \qquad g(x_0) = \frac{1}{\sqrt{4}} = \frac{1}{2}$$

$$g'(x) = -\frac{1}{2}(x-3)^{-3/2} \qquad g''(x) = \frac{3}{2\cdot 2}(x-3)^{-5/2} \qquad g'''(x) = -\frac{3\cdot 5}{2^3}(x-3)^{-7/2}$$

$$g^{(m)}(x) = (-1)^m \frac{(2m-1)!!}{2^m}(x-3)^{-\frac{2m+1}{2}} \quad \checkmark\checkmark$$

$$g^{(m)}(x_0) = (-1)^m \frac{(2m-1)!!}{2^m}\left(\frac{1}{2}\right)^{2m+1} = \frac{1}{2}(-1)^m (2m-1)!! \cdot \frac{1}{2^m}\left(\frac{1}{4}\right)^m \checkmark$$

$$\frac{g^{(m)}(x_0)}{m!} = \frac{1}{2}(-1)^m \frac{(2m-1)!!}{(2m)!!}\left(\frac{1}{4}\right)^m$$

Taylorova řada:

$$\frac{1}{2} + \frac{1}{2}\sum_{m=1}^{\infty}(-1)^m \frac{(2m-1)!!}{(2m)!!}\left(\frac{1}{4}\right)^m (x-7)^m \quad \checkmark\checkmark$$

Poloměr konvergence:

$$R^{-1} = \lim_{m\to\infty} \frac{(2m+1)!!}{(2m+2)!!}\frac{4^m}{4^{m+1}}\frac{(2m)!!}{(2m-1)!!} = \lim_{m\to\infty}\frac{1}{4}\frac{2m+1}{2m+2} = \frac{1}{4} \Rightarrow R = 4$$

$\checkmark$

Interval konvergence:

$$I = (3; 11) \leftarrow \checkmark \text{ (krajní body)}$$

Konvergence v krajních bodech: $\sum_{m=1}^{\infty}(\pm)^m \frac{(2m-1)!!}{(2m)!!}$

(pokud student nepracuje se správnými krajními body, neziskám žádné body)

Raabe: $\checkmark\checkmark$

$$\lim_{m\to\infty} m\left(1 - \left|\frac{a_{m+1}}{a_m}\right|\right) = \lim_{m\to\infty} m\left(1 - \frac{(2m+1)!!}{(2m+2)!!}\cdot\frac{(2m)!!}{(2m-1)!!}\right) =$$

$$= \lim_{m\to\infty} m\left(1 - \frac{2m+1}{2m+2}\right) = \lim_{m\to\infty} m\left(\frac{2m+2-2m-1}{2m+2}\right) =$$

$$= \lim_{m\to\infty} m\,\frac{1}{2m+2} = \frac{1}{2} \in (0; 1)$$

Obor konvergence:

$$O = (3; 11\rangle \quad \checkmark$$

$\rho(\vec{x},\vec{y}) = 3\sqrt{|x_1-y_1|} + 2\sqrt{|x_2-y_2|}$ $\quad U_6(\vec{0}) = ?$ 5b

Hranica okolia: $3\sqrt{|x_1|-0} + 2\sqrt{|x_2|-0} = 6$

$3\sqrt{|x_1|} + 2\sqrt{|x_2|} = 6 \quad / x_1 \geq 0 \wedge y_2 \geq 0$

$x_2 = \left(3 - \frac{3}{2}\sqrt{x_1}\right)^2$

$x_1 = 0$ $\quad\quad$ $x_2 = 0$

$\Downarrow$ $\quad\quad$ $6 - 3\sqrt{x_1} = 0$

$x_2 = 9$ $\quad\quad$ $x_1 = 4$

Konvexnosť vs. konkávnosť:

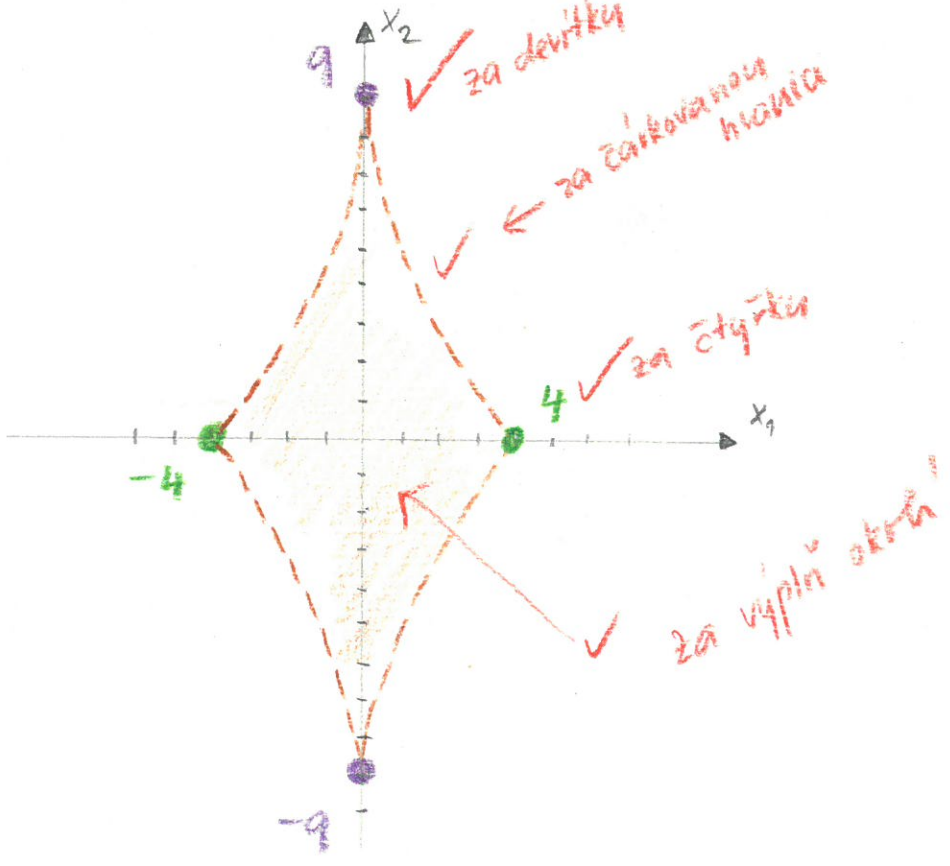
$g(x) = \left(3 - \frac{3}{2}\sqrt{x}\right)^2$

$g'(x) = 2\left(3 - \frac{3}{2}\sqrt{x}\right) \cdot \left(-\frac{3}{2}\right) \cdot \frac{1}{2}\frac{1}{\sqrt{x}} =$

$= -\frac{3}{2}\left(\frac{3}{\sqrt{x}} - \frac{3}{2}\right)$

$g''(x) = \frac{9}{2} \cdot \frac{1}{2}\frac{1}{x^{3/2}} > 0 \Rightarrow g(x)$ je v $(0,4)$ konvexná!

✓ za konvexnosť hranice $V(\mathbb{R}^+)^2$



9 bodů

$$y'' + 4y' + 4y = e^{-2x}\left(1 + \frac{1}{x}\right)$$

```
In[4]:= DSolve[y''[x] + 4 y'[x] + 4 y[x] == Exp[-2 x] + Exp[-2 x] / x, y[x], x] // FullSimplify
Out[4]= {{y[x] → 1/2 E^(-2 x) (2 C[1] + x (-2 + x + 2 C[2]) + 2 x Log[x])}}
```

$\lambda^2 + 4\lambda + 4 = (\lambda+2)^2 \quad \Rightarrow \quad F_0 = \{e^{-2x};\ xe^{-2x}\}$ ✓

$W = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & (1-2x)e^{-2x} \end{vmatrix} = e^{-4x}(1 - 2x + 2x) = e^{-4x} \neq 0$ ✓

$D_1 = \begin{vmatrix} 0 & xe^{-2x} \\ e^{-2x}\left(1+\frac{1}{x}\right) & (1-2x)e^{-2x} \end{vmatrix} = -e^{-4x}(x+1)$

$D_2 = \begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & e^{-2x}\left(1+\frac{1}{x}\right) \end{vmatrix} = e^{-4x}\left(1+\frac{1}{x}\right)$

$f_1(x) = \frac{D_1}{W} = -x - 1$ ✓ $\quad \Rightarrow \quad F_1(x) = -\frac{x^2}{2} - x + \alpha$ ✓

$f_2(x) = \frac{D_2}{W} = 1 + \frac{1}{x}$ ✓ $\quad \Rightarrow \quad F_2(x) = x + \ln x + \beta$ ✓

$$y(x) = \alpha e^{-2x} + \beta x e^{-2x} - \frac{1}{2}x^2 e^{-2x} - xe^{-2x} + x^2 e^{-2x} + e^{-2x}\ln(x)$$ ✓

↖ tento člen je už zahrnut ve členu $\beta x e^{-2x}$

Optimalizace výsledku:

$$y(x) = \alpha e^{-2x} + \beta x e^{-2x} + \frac{1}{2}e^{-2x}(x^2 + 2\ln x)$$

↖ ✓ výsledek po optimalizaci!

$\mathrm{Dom}(y) = (0; +\infty)$ ✓

Volba mocninné řady: $\sum_{n=1}^{\infty} n^2 x^n = f(x)$ ✓

$f\left(\frac{1}{5}\right) = \sum_{n=1}^{\infty} n^2 \left(\frac{1}{5}\right)^n$

$\frac{1}{5} \in O \Leftarrow \lim_{n\to\infty} \frac{(n+1)^2}{5^{n+1}} \cdot \frac{5^n}{n^2} = \frac{1}{5} < 1$ ✓

Hledání součtu:

$h(x) = \frac{f(x)}{x} = \sum_{n=1}^{\infty} n^2 x^{n-1}$

$H(x) = \int h(x)\,dx = \sum_{n=1}^{\infty} n x^n$

$g(x) = \frac{H(x)}{x} = \sum_{n=1}^{\infty} n x^{n-1}$ ✓

$G(x) = \int g(x)\,dx = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$ ✓ $\Rightarrow R = 1 \Rightarrow I = (-1, 1)$ & $\frac{1}{5} \in I$

nebo

$g(x) = \frac{dG}{dx} = \frac{1-x+x}{(1-x)^2} = \frac{1}{(1-x)^2}$

$H(x) = \frac{x}{(1-x)^2}$ ✓

$h(x) = \frac{dH}{dx} = \frac{(1-x)^2 + 2x(1-x)}{(1-x)^4} = \frac{1-x+2x}{(1-x)^3} = \frac{1+x}{(1-x)^3}$

$f(x) = \frac{x(1+x)}{(1-x)^3}$ ✓✓

$\Rightarrow \sum_{n=1}^{\infty} \frac{n^2}{5^n} = f\left(\frac{1}{5}\right) = \frac{\frac{1}{5}\cdot\frac{6}{5}}{\left(\frac{4}{5}\right)^3} = \frac{6}{5^2} \cdot \frac{5^3}{16 \cdot 4} = \frac{15}{32}$ ✓✓

musí být uvedeno OK a v základním tvaru!

$$y = e^x \sin 2x + \frac{1}{2} x e^x \sin 2x$$

$$y = e^x \sin 2x \left(1 + \frac{x}{2}\right); \quad x \in \mathbb{R}$$

2. Nalezněte implicitní tvar řešení rovnice

$$(y - 2x) = y'(y - x)$$

Diskutujte, co řešení představuje. 8 bodů

$$y'(y-x) = y - 2x \quad / \frac{1}{x}$$

$$y'\left(\frac{y}{x} - 1\right) = \frac{y}{x} - 2$$

$z = \frac{y}{x} \Rightarrow y = zx \Rightarrow y' = z'x + z$

$$(z'x + z)(z-1) = z - 2$$

$$z'x(z-1) + z^2 - z = z - 2$$

$$z'x(z-1) = -(z^2 - 2z + 2)$$

$$z' \frac{z-1}{z^2-2z+2} = -\frac{1}{x}$$

$$\frac{1}{2} \frac{2z-2}{z^2-2z+2} dz = -\frac{1}{x} dx$$

$$\ln \sqrt{z^2-2z+2} = \ln \frac{1}{|x|} C$$

$$z^2 - 2z + 2 = \frac{C}{x^2}$$

$$C = x^2\left(\frac{y^2}{x^2} - 2\frac{y}{x} + 2\right)$$

$$C = y^2 - 2xy + 2x^2 = (x+y)^2 + x^2$$

– jde o elipsu, jejíž osa není rovnoběžná ani s osou x ani s osou y

Ukažte, že matice

$$A = \begin{pmatrix} \langle \vec{u} | \vec{u} \rangle & \langle \vec{w} | \vec{u} \rangle \\ \langle \vec{u}, \vec{w} \rangle & \langle \vec{w} | \vec{w} \rangle \end{pmatrix}$$

5b (1 extra)

je za všech okolností pozitivně definitní! Kdy A PD není?

Směr

Schwarz-Cauchy-Buňakovský

$$\text{SCB:} \quad \frac{|\langle \vec{u} | \vec{v} \rangle|}{\sqrt{\langle u | u \rangle} \cdot \sqrt{\langle \vec{w} | \vec{w} \rangle}} \leq 1$$

axiomy normy

Δ_1:

$$\langle \vec{u} | \vec{u} \rangle = \|\vec{u}\|^2 > 0 \qquad \vec{u} \neq \vec{o}$$

Sylvester. kritérium:

$$\Delta_2 = = \|\vec{u}\|^2 \cdot \|\vec{w}\|^2 - \langle \vec{u} | \vec{v} \rangle \cdot \langle \vec{u} | \vec{w} \rangle^* =$$

$$= \|\vec{u}\|^2 \cdot \|w\|^2 - |\langle u | \vec{v} \rangle|^2 \geq$$

$$\geq \|\vec{u}\|^2 \cdot \|\vec{w}\| - \|u\|^2 - \|\vec{w}\| \geq 0$$

$$\boxed{\begin{array}{l} \text{pro LZ je } A \geq 0 \\ \text{pro LN je } A > 0 \end{array}}$$

$\lim_{n\to\infty} x^n e^{-nx} = 0$ ✓ $\Leftarrow$ $\lim_{n\to\infty} \frac{x^n}{e^{nx}} \overset{l'H}{=} \lim_{n\to\infty} \frac{n!}{n^n e^{-nx}} = 0$

$$\rho^2(x^n e^{-nx}, 0) = \|x^n e^{-nx} - 0\|^2 = \langle x^n e^{-nx} \mid x^n e^{-nx} \rangle =$$

$$= \int_0^\infty x^{2n} \cdot e^{-2nx} \cdot \underbrace{e^{-x}}_{w(x)} dx \checkmark = \int_0^\infty x^{2n} e^{-(2n+1)x} dx =$$

$$= \frac{(2n-1)!}{(2n+1)^{2n+1}} \checkmark$$

$\sum \frac{(2n-1)!}{(2n+1)^{2n+1}}$ konverguje $\overset{\text{nutná podmínka}}{\Rightarrow}$ $\lim_{n\to\infty} \frac{(2n-1)!}{(2n+1)^{2n+1}} = 0$ ✓

podílové kritérium:

$$\lim_{n\to\infty} \frac{(2n+1)!}{(2n+3)^{2n+3}} \cdot \frac{(2n+1)^{2n+1}}{(2n-1)!} = \lim_{n\to\infty} \frac{(2n+1)^{2n+2}}{(2n+3)^{2n+3}} = \lim_{n\to\infty} \frac{1}{2n+3} \cdot \lim_{n\to\infty} \left(\frac{2n+1}{2n+3}\right)^{2n+2} =$$

$$= \lim_{n\to\infty} \frac{1}{2n+3} \cdot \lim_{n\to\infty} \left(\frac{2n+1}{2n+3}\right)^2 \cdot \left[\lim_{n\to\infty} \left(1 - \frac{2}{2n+3}\right)^n\right]^2 =$$

$$= 0 \cdot 1 \cdot$$

$\rho^2(x^n e^{-nx}, 0) = \ldots$

$\frac{1}{3^3} \Leftarrow n=1$ $\quad \frac{1}{3^3} > \frac{7!}{9^9}$ (1)

$\frac{3!}{5^5} \Leftarrow n=2$ $\quad \frac{3!}{5^5} > \frac{7!}{9^9}$ (2)

$\frac{5!}{7^7} \Leftarrow n=3$ $\quad \frac{5!}{7^7} > \frac{7!}{9^9}$ (3)

$\frac{7!}{9^9} \Leftarrow n=4$ $\quad \frac{7!}{9^9} = \frac{7!}{9^9}$ (4)

$\frac{9!}{11^{11}} \Leftarrow n=5$ $\quad \frac{9!}{11^{11}} < \frac{7!}{9^9}$

ČTYŘI ✓✓

[A4] Nalezněte maximální řešení diferenciální rovnice

$$y''' - 3y' - 2y = 6e^{-x} - 4xe^{x}$$

$$(\lambda^3 - 3\lambda - 2) : (\lambda + 1) = \lambda^2 - \lambda - 2$$
$$-\lambda^3 - \lambda^2$$
$$-\lambda^2 - 3\lambda - 2$$
$$\lambda^2 + \lambda$$
$$-2\lambda - 2$$

$$\lambda_1 = -1$$
$$\lambda_{2,3} = \frac{1 \pm \sqrt{1+4\cdot 2}}{2} = \frac{1 \pm 3}{2} = \begin{cases} 2 \\ -1 \end{cases}$$

$$FŘ = \{e^{-x}, xe^{-x}, e^{2x}\}$$

- $L(y_1) = 6e^{-x}$ — -1 je dvojnásobným kořenem
  $\Rightarrow$ předpokládaný tvar: $y_1 = ax^2e^{-x}$

$$y_1' = a(2x - x^2)e^{-x}$$
$$y_1'' = a(2 - 4x + x^2)e^{-x}$$
$$y_1'' = a(2 - 2x - 2x + x^2)e^{-x}$$
$$y_1''' = a(-4 + 2x - 2 + 4x - x^2)e^{-x}$$
$$y_1''' = a(-6 + 6x - x^2)e^{-x}$$

Dosazení: $-6a + 6ax - ax^2 - 6ax + 3ax^2 - 2ax^2 = 6$

$$a = -1$$

$$y_1 = -x^2e^{-x}$$

- $L(y_2) = -4xe^{x}$ $\qquad y_2 = (ax + b)e^{x}$

$$y_2' = (a + b + ax)e^{x}$$
$$y_2'' = (2a + b + ax)e^{x}$$
$$y_2''' = (3a + b + ax)e^{x}$$

Dosazení: $3a + b + ax - 3a - 3b - 3ax - 2ax - 2b = -4x$

$$-4b - 4ax = -4x$$

$$(a, b) = (1, 0)$$

$$y_2 = xe^{x}$$

Celkové řešení: $y(x) = c_1e^{-x} + c_2xe^{-x} + c_3e^{2x} - x^2e^{-x} + xe^{x}; \quad x \in \mathbb{R}$

8 bodů

$t = x+4$

$$\sum_{n=1}^{\infty} \left(\frac{7}{n^2} + \frac{2}{n}\right) t^n$$

- $$R^{-1} = \lim_{n\to\infty} \frac{\frac{7}{(n+1)^2} + \frac{2}{n+1}}{\frac{7}{n^2} + \frac{2}{n}} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} \cdot \frac{7 + 2(n+1)}{7 + 2n} = 1$$

$$\Rightarrow R_t = 1$$

- $t = x+4 = 1$ řada $\sum_{n=1}^{\infty} \left(\frac{7}{n^2} + \frac{2}{n}\right)$

$\forall n \in \mathbb{N}$: $\frac{2}{n} < \frac{7}{n^2} + \frac{2}{n}$ $\Rightarrow$ řada diverguje podle srovnávacího kritéria, neboť $\sum \frac{2}{n}$ je divergentní

- $t = x+4 = -1$ řada $\sum_{n=1}^{\infty} (-1)^n \left(\frac{7}{n^2} + \frac{2}{n}\right)$

– obě řady $\sum_{n} (-1)^n \frac{7}{n^2}$ i $\sum_{n=1}^{\infty} (-1)^n \frac{2}{n}$ konvergují

$$O = \langle -5, -3)$$

10 b

$$s(x) = \sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!} \qquad s(0) = 1$$

$$s'(x) = \sum_{n=1}^{\infty} \frac{x^{4n-1}}{(4n-1)!} \qquad s'(0) = 0$$

$$s''(x) = \sum_{n=1}^{\infty} \frac{x^{4n-2}}{(4n-2)!} \qquad s''(0) = 0$$

$$s'''(x) = \sum_{n=1}^{\infty} \frac{x^{4n-3}}{(4n-3)!} \qquad s'''(0) = 0$$

$$s^{(4)}(x) = \sum_{n=1}^{\infty} \frac{x^{4n-4}}{(4n-4)!} = \Big| \, m = n-1 \, \Big| = \sum_{m=0}^{\infty} \frac{x^{4m}}{(4m)!} = s(x)$$

Formulace (explicitní) Cauchyovy úlohy ✓

$$s^{(4)}(x) - s(x) = 0$$

$$\lambda^4 - 1 \equiv (\lambda^2 - 1)(\lambda^2 + 1) \quad \Rightarrow \quad \lambda \in \{\pm 1, \pm i\}$$

$$\Rightarrow \mathcal{F}_S = \{\operatorname{ch} x, \operatorname{sh} x, e^x, e^{-x}\}$$

$$s(x) = C_1 \operatorname{ch} x + C_2 \operatorname{sh} x + C_3 e^x + C_4 e^{-x} \qquad (C_1, C_2, C_3, C_4) = ?$$

$$s'(x) = -C_1 \operatorname{sh} x + C_2 \operatorname{ch} x + C_3 e^x - C_4 e^{-x}$$

$$s''(x) = -C_1 \operatorname{ch} x - C_2 \operatorname{sh} x + C_3 e^x + C_4 e^{-x}$$

$$s'''(x) = C_1 \operatorname{sh} x - C_2 \operatorname{ch} x + C_3 e^x - C_4 e^{-x}$$

$$\left.\begin{array}{l} C_1 + C_3 + C_4 = 1 \\ C_2 + C_3 - C_4 = 0 \\ -C_1 + C_3 + C_4 = 0 \\ -C_2 + C_3 - C_4 = 0 \end{array}\right\} \; C_1 = \frac{1}{2} \; \Big\} \; C_2 = 0 \qquad \left.\begin{array}{l} C_3 + C_4 = \frac{1}{2} \\ C_3 - C_4 = 0 \end{array}\right\} \; (C_3, C_4) = (\tfrac{1}{4}, \tfrac{1}{4})$$

$$(C_1, C_2, C_3, C_4) = (\tfrac{1}{2}, 0, \tfrac{1}{4}, \tfrac{1}{4})$$

$$s(x) = \frac{1}{2} \operatorname{ch} x + \frac{1}{4} e^x + \frac{1}{4} e^{-x}$$

$$\sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!} = \frac{1}{2}(\operatorname{ch} x + \cos x) \qquad 0 = \operatorname{Dom}(s) = \mathbb{R}$$

$$\varphi(x) = \sum_{n=1}^{\infty} \frac{2x}{n^4 + x^4}$$

- primitivní funkce
- její G

11b

$$\Phi(x) = \int_0^x \varphi(t)\,dt = \int_0^x \sum_{n=1}^{\infty} \frac{2t}{n^4+t^4}\,dt$$

- nutno rozhodnout, zda lze provést záměnu

$$\left(\frac{2x}{n^4+x^4}\right)' = 2\,\frac{n^4 + x^4 - x\cdot 4x^3}{(n^4+x^4)^2} = 2\,\frac{n^4 - 3x^4}{(n^4+x^4)^2} = 0$$

$$x = \frac{1}{3^{1/4}}\,n$$

$$G_n = \varphi\left(\frac{n}{3^{1/4}}\right) = 2\,\frac{1}{3^{1/4}}\,\frac{n}{n^4 + \frac{1}{3}n^4} = \frac{2}{3^{1/4}}\,\frac{3}{3n^3+n^3} = \frac{2\cdot 3^{3/4}}{4n^3}$$

$$G_n = \frac{3^{3/4}}{2n^3}$$

$$\left|\frac{2x}{n^4+x^4}\right| \le \frac{3^{3/4}}{2n^3} = G_n$$

Řada $\sum_n G_n$ zjevně konverguje $\Rightarrow$ $\varphi(x)$ SK na $\mathbb{R}$

$\Rightarrow$ záměna je možná

$$\Phi(x) = \sum_{n=1}^{\infty} \int_0^x \frac{2t}{n^4+t^4}\,dt = \left| \begin{matrix} \tau = t^2 \\ d\tau = 2t\,dt \end{matrix} \right| = \sum_{n=1}^{\infty} \int_0^{x^2} \frac{1}{n^4+\tau^2}\,d\tau =$$

$$= \sum_{n=1}^{\infty} \int_0^{x^2} \frac{1/n^4}{1+\left(\frac{\tau}{n^2}\right)^2}\,d\tau = \sum_{n=1}^{\infty} \left[\frac{1}{n^2}\operatorname{arcotg}\frac{\tau}{n^2}\right]_0^{x^2} =$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2}\operatorname{arcotg}\frac{x^2}{n^2}$$

$$\left|\frac{1}{n^2}\operatorname{arcotg}\frac{x^2}{n^2}\right| \le \frac{1}{2n^2} \quad \Rightarrow \quad G = \mathbb{R}$$

ze srovnávacího (Weierstrassova) kritéria: $\sum_{n=1}^{\infty} \frac{1}{n^2}\operatorname{arcotg}\frac{x^2}{n^2}$ SK na G

zdůvodnit!