9b

$$G(b) = \int_0^\infty e^{-ax} \frac{\cos^2(cx) - \cos^2(bx)}{x}\,dx$$

$$\frac{dG}{db} = \int_0^\infty e^{-ax} \frac{2\cos(bx)\cdot\sin(bx)\cdot x}{x}\,dx = \int_0^\infty e^{-ax}\cdot \sin(2bx)\,dx$$

$$\left.\begin{aligned} \int_0^\infty e^{-(a+i\omega)x}\,dx &= \left[-\frac{e^{-(a+i\omega)x}}{a+i\omega}\right]_0^\infty = \frac{1}{a+i\cdot\omega} = \frac{a-i\omega}{a^2+\omega^2} \\ \int_0^\infty e^{-(a+i\omega)x}\,dx &= \int_0^\infty e^{-ax}\cos(\omega x)\,dx - i\int_0^\infty e^{-ax}\sin(\omega x)\,dx \end{aligned}\right\} \Rightarrow \int_0^\infty e^{-ax}\sin(\omega x)\,dx = \frac{\omega}{a^2+\omega^2}$$

$$\frac{dG}{db} = \frac{2b}{a^2+4b^2}$$

$$G(b) = \int \frac{2b}{a^2+4b^2}\,db = \left| \begin{aligned} u &= a^2+4b^2 \\ du &= 8b\,db \end{aligned} \right| = \frac{1}{4}\int \frac{du}{u}\,du =$$

$$= \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln(a^2+4b^2) + C$$

$$G(b=c) \overset{!}{=} 0 \Rightarrow \frac{1}{4}\ln(a^2+4c^2) + C \overset{!}{=} 0 \Rightarrow C = -\frac{1}{4}\ln(a^2+4c^2)$$

Závěr:

$$G(b) = \frac{1}{4}\ln\frac{a^2+4b^2}{a^2+4c^2}$$

Předpoklady pro možnost derivovat podle parametru:

α) $G(b)$ jistě konverguje pro $b=c$, a sice $G(b=c)=0$

β) měřitelný integrand (jedná se o spojitou funkci) $\leftarrow$ dokonce na $\langle 0;+\infty)$

γ) integrabilní majoranta k derivaci nezávislá na b

$$\underbrace{|e^{-ax}\cdot\sin(2bx)| \leq e^{-ax} \in \mathcal{L}(\mathbb{R}^+)}$$

$$\Uparrow$$

$$\underbrace{\int_0^{+\infty} e^{-ax}\,dx = \left[-\frac{1}{a}e^{-ax}\right]_0^\infty = \frac{1}{a} \in \mathbb{R}}$$

Rozhodujeme vlastně o konvergenci integrálu $\int_{(0,1)} x^a \, dx$. 5b.

Protože uvažujeme míru $\lambda(x)$ lze tento integrál chápat jako $\int_0^1 x^a \, dx$.

Pak snadno:

$$a > -1 \Rightarrow \int_0^1 x^a \, dx = \left[\frac{x^{a+1}}{a+1} \right]_0^1 = |a+1>0| = \frac{1}{a+1} \left.\right\} \begin{array}{l} a \geq 0 \checkmark \\ a \in (-1,0) \checkmark\checkmark \end{array}$$

$$a = -1 \Rightarrow \int_0^1 x^a \, dx = \int_0^1 \frac{1}{x} \, dx = \left[\ln x \right]_0^1 = +\infty \checkmark$$

$$a < -1 \Rightarrow \int_0^1 x^a \, dx = \left[\frac{x^{a+1}}{a+1} \right]_0^1 = |a+1<0| = +\infty \checkmark$$

$\Rightarrow$ Výsledek $\Theta(x)\Theta(1-x)x^a \in \mathcal{L}(\mathbb{R}, \lambda)$ platí pro $a > -1$

11 b

$z = z(x, y, u)$ ✓ explicitní uvedení, o jakou funkci jde

Existenční podmínka: $\frac{\partial F}{\partial z} = 3z^2 + 6uz \neq 0$ ✓ !

Nutná podmínka pro lokální extrém funkce $z(x,y,u)$: $\operatorname{grad} z \overset{!}{=} \vec{0}$

$$\Rightarrow \left(\frac{\partial F}{\partial x}; \frac{\partial F}{\partial y}; \frac{\partial F}{\partial z} \right) \overset{!}{=} \vec{0} \Leftarrow \frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}; \frac{\partial z}{\partial y} = \dots \text{ atd}$$ ✓

viz zadání!

$$\left.\begin{cases} 8x + 24y = 0 \\ 24x + 4y + 68 = 0 \\ 3z^2 = 3u^2 \\ F(x,y,z,u) = 0 \end{cases}\right\} \Rightarrow$$ ✓

$x = -3, y = -1; z = u > 0$

$u^3 + 3u^3 - u^3 + 36 - 72 + 2 + 68 = 37$

$3u^3 = 3 \Rightarrow u = 3$

Stacionární bod: $\vec{a} = (a_x, a_y, a_u) = (-3, 1, 1)$ ✓

Příslušný generující bod: $\vec{\lambda} = (a_x, a_y, a_u, a_z) = (-3, 1, 1, 1)$ ✓

Kontrola existenční podmínky:

$$\frac{\partial F}{\partial z}(\vec{\lambda}) = 3 + 6 = 9 \neq 0$$ ✓

Hessova matice generující funkce F:

$$H_F = \begin{pmatrix} \boxed{\begin{matrix} 8 & 24 & 0 \\ 24 & 4 & 0 \\ 0 & 0 & * \end{matrix}} & \begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix} \\ \begin{matrix} 0 & 0 & 6z \end{matrix} & 3z^2 + 6uz \end{pmatrix}$$

$* = -6u$

Hessova matice implicitní funkce ve stacionárním bodě:

$$H_z = -\frac{1}{9} \begin{pmatrix} 8 & 24 & 0 \\ 24 & 4 & 0 \\ 0 & 0 & -6u \end{pmatrix}\Bigg|_{\vec{a}} = -\frac{1}{9} \begin{pmatrix} 8 & 24 & 0 \\ 24 & 4 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

Druhý totální diferenciál implicitní funkce $z(x,y,u)$ v bodě $\vec{a}$:

(upravený až na násobení kladným číslem) ✓✓

$$d^2 z_{\vec{a}}(dx, dy, dz) = -4dx^2 - 2dy^2 - 24dx\,dy + 3du^2 =$$

$$= -2\left[dy^2 + 2dx^2 + 12dx\,dy\right] + 3u^2 =$$

$$= -2(dy + 6dx)^2 + 34dx^2 \cdot 2 + 3u^2 \lessgtr 0$$ ✓

$\Rightarrow$ ve stacionárním bodě $\vec{a} = (-3, 1, 1)$ extrém nenastává $\Rightarrow$

$\Rightarrow$ jedná se o sedlový bod ✓

❹ (5 bodů) Heineovou větou rozhodněte, zda má funkce

$$h(x,y) = \begin{cases} \frac{y(x+3)}{y^2+(x+3)^2} + 3(x-1) & \dots \quad (x,y) \neq (-3,0), \\ -12 & \dots \quad (x,y) = (-3,0), \end{cases}$$

limitu v bodě $(-3,0)$. Své tvrzení podrobně komentujte!

Pomocný výpočet: $M_k = \{(x,y)\in\mathbb{R}^2 : y = k\cdot(x+3)\};\ k\in\mathbb{R};\ (-3,0)\in M_k$

$$\lim_{\substack{(x,y)\to\vec{0} \\ (x,y)\in M_k}} h(x,y) = \lim_{x\to -3}\left[\frac{k(x+3)^2}{k^2(x+3)^2+(x+3)^2} + 3(x-1)\right] = \frac{k}{k^2+1} - 12$$

Pro aplikaci Heineovy věty hledáme posloupnost vektorů $\vec{\omega}_n = (x_n, y_n)$, která konverguje k bodu $(-3,0)$, ale nepotíná ho, pro kterou $\lim\limits_{n\to\infty} h(\vec{\omega}_n) \neq -12$ ✓

Zvolme (na základě pomocného výpočtu):

$$x_n = -3 + \frac{1}{n} \quad \& \quad y_n = k(x_n+3) = \frac{k}{n}$$ ✓

Zjevně:

$$(x_n, y_n) \xrightarrow{n\to\infty} (-3,0)$$ ✓

$$\forall n\in\mathbb{N}: \quad \left(-3+\frac{1}{n};\ \frac{k}{n}\right) \neq (-3,0)$$ ✓

$$\lim_{n\to\infty} h(\vec{\omega}_n) = \lim_{n\to\infty}\left[\frac{\frac{k}{n}\cdot\frac{1}{n}}{\frac{k^2}{n^2}+\frac{1}{n^2}} + 3\left(-3+\frac{1}{n}-1\right)\right] = \frac{k}{k^2+1} - 12$$

Stačí volit např. $k = 1$. Pak $\lim\limits_{n\to\infty} h(\vec{\omega}_n) = h\left(-3+\frac{1}{n};\ \frac{1}{n}\right) = \frac{1}{2} - 12 = -\frac{23}{2} \neq -12$ ✓

a proto $\lim\limits_{(x,y)\to(-3,0)} h(x,y)$ existovat nemůže! ✓

$$\underbrace{\frac{x^2y^2}{a^4}+\frac{y^2z^2}{b^4}+\frac{x^2z^2}{c^4}=1}_{\text{generující rovnice pro implicitní funkci } z=z(x,y)} \quad \& \quad \frac{x_0^2y_0^2}{a^4}+\frac{y_0^2z_0^2}{b^4}+\frac{x_0^2z_0^2}{c^4}=1$$

$(x_0, y_0, z_0) \in P$

& $\vec{x}=(x_0,y_0,z_0)$ je generující bod

Podmínka nekritičnosti (tj. garance existence implicitní funkce)

$$\frac{\partial F}{\partial z} = 2z\frac{y^2}{b^4}+2z\frac{x^2}{c^4}\neq 0 \Rightarrow z_0\neq 0 \wedge \frac{y_0^2}{b^4}+\frac{x_0^2}{c^4}\neq 0$$

$\Rightarrow x_0 \neq 0$, $y_0\neq 0$

$$\frac{\partial z}{\partial x} = -\frac{x}{z}\,\frac{\frac{y^2}{a^4}+\frac{z^2}{c^4}}{\frac{y^2}{b^4}+\frac{x^2}{c^4}} \quad \& \quad \frac{\partial z}{\partial y} = -\frac{y}{z}\,\frac{\frac{x^2}{a^4}+\frac{z^2}{b^4}}{\frac{y^2}{b^4}+\frac{x^2}{c^4}}$$

Rovnice tečné roviny:

$$z-z_0 = -\frac{x_0}{z_0}\,\frac{\frac{y_0^2}{a^4}+\frac{z_0^2}{c^4}}{\frac{y_0^2}{b^4}+\frac{x_0^2}{c^4}}(x-x_0) - \frac{y_0}{z_0}\,\frac{\frac{x_0^2}{a^4}+\frac{z_0^2}{b^4}}{\frac{y_0^2}{b^4}+\frac{x_0^2}{c^4}}(y-y_0)$$

$$(zz_0-z_0^2)\left(\frac{y_0^2}{b^4}+\frac{x_0^2}{c^4}\right) + x_0\left(\frac{y_0^2}{a^4}+\frac{z_0^2}{c^4}\right)(x-x_0) + y_0\left(\frac{x_0^2}{a^4}+\frac{z_0^2}{b^4}\right)(y-y_0) = 0$$

$$\frac{y_0^2z_0}{b^4}z + \frac{x_0^2z_0}{c^4}z + \frac{x_0y_0^2}{a^4}x + \frac{x_0z_0^2}{c^4}x + \frac{y_0x_0^2}{a^4}y + \frac{y_0z_0^2}{b^4}y =$$

$$= \underbrace{\frac{y_0^2z_0^2}{b^4} + \frac{x_0^2z_0^2}{c^4} + \frac{x_0^2z_0^2}{c^4} + \frac{x_0^2y_0^2}{a^4} + \frac{x_0^2y_0^2}{a^4} + \frac{y_0^2z_0^2}{b^4}}_{=2}$$

nějaké úpravy

$$\underbrace{\frac{\overbrace{xx_0y_0^2+yy_0x_0^2}^{x_0y_0(xy_0+x_0y)}}{a^4} + \frac{zz_0y_0^2+yy_0z_0^2}{b^4} + \frac{zz_0x_0^2+xx_0z_0^2}{c^4} = 2}_{\text{elegantní tvar}}$$

Pozn.

Pokud by (x_0, y_0, z_0) byl kritický (s ohledem na $z=z(x,y)$), pak by stačilo volit např. $y=y(x,z)$. Takže $\underbrace{\text{výsledek platí } \forall (x_0,y_0,z_0)\in P}_{\text{extra bod *}}$!

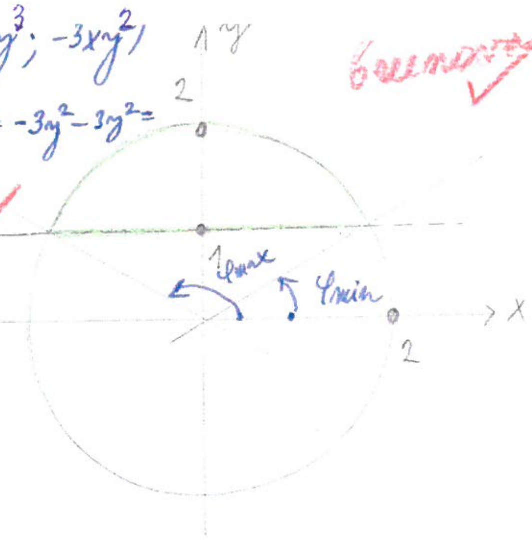
10b

$F(x,y) = (y^3; -3xy^2)$

Greenova věta

$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -3y^2 - 3y^2 =$

$= -6y^2$



$\underbrace{\varphi_{min} = \frac{\pi}{6} \quad \& \quad \varphi_{max} = \frac{5\pi}{6}}$

$x = \rho\cos\varphi$

$y = \rho\sin\varphi$

$x^2 + y^2 \leq 4$ | $y \geq 1$

$\rho^2 \leq 4$ | $\rho\sin\varphi \geq 1$

$\rho \leq 2$ | $\rho \geq \frac{1}{\sin\varphi}$

$$6\int_{-2} y^2\, d(x,y) = 6\int_{\pi/6}^{5\pi/6}\int_{1/\sin\varphi}^{2} \rho^2 \cdot \sin^2\varphi \cdot \rho\, d\rho\, d\varphi =$$

(jsou-li chybné meze, dále se neopravuje)

$$= 6\int_{\pi/6}^{5\pi/6} \sin^2\varphi \left[\frac{1}{4}\rho^4\right]_{1/\sin\varphi}^{2} d\varphi = 6\cdot\frac{1}{4}\int_{\pi/6}^{5\pi/6} \sin^2(\varphi)\cdot\left(16 - \frac{1}{\sin^4\varphi}\right) d\varphi =$$

$$= 6\cdot 4\int_{\pi/6}^{5\pi/6} \sin^2(\varphi)\, d\varphi - \frac{6}{4}\int_{\pi/6}^{5\pi/6} \frac{1}{\sin^2(\varphi)}\, d\varphi =$$

$$= 6\cdot 4\int_{\pi/6}^{5\pi/6} \underbrace{\frac{1-\cos(2\varphi)}{2}}\, d\varphi + \frac{6}{4}\left[\underbrace{\cot(\varphi)}\right]_{\pi/6}^{5\pi/6} =$$

$$= 12\left[\varphi - \frac{1}{2}\sin(2\varphi)\right]_{\pi/6}^{5\pi/6} + \frac{6}{4}\left(-\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} - \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) =$$

$$= 6\left(\frac{4}{3}\pi - \left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) + \frac{1}{4}\left(-2\sqrt{3}\right)\right) = 6\left(\frac{4}{3}\pi + \sqrt{3} - \frac{\sqrt{3}}{2}\right) = \left(\frac{4}{3}\pi + \frac{\sqrt{3}}{2}\right)\cdot 6 =$$

$$= 1\pi + 3\sqrt{3}$$

← správný výsledek

9b

❶ (bodů) Vypočítejte integrál z vektorové funkce $(6x^2y^3, x^3y^2)$ po křivce

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^4 = \frac{2xy}{ab}.$$

$\vec{G}(x,y) = (G_1(x,y); G_2(x,y)) = $
$= (6x^2y^3; x^3y^2)$

Křivka je jednoduchá, uzavřená a hladká regulární $\Rightarrow$ půjde užít Greenovu větu

$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = 3x^2y^2 - 18x^2y^2 = -15x^2y^2$$

Plocha ohraničená křivkou: $S = \{(x,y)\in\mathbb{R}^2:\ \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^4 \leq \frac{2xy}{ab}\}$

$$\left.\begin{aligned} x &= a\rho\cos\varphi \\ y &= b\rho\sin\varphi \end{aligned}\right\} \longrightarrow \quad \rho^8 \leq 2\rho^2\cos(\varphi)\cdot\sin(\varphi)$$

$$\rho^6 \leq \sin(2\varphi) \Rightarrow \varphi\in(0;\tfrac{\pi}{2})\cup(\pi;\tfrac{3\pi}{2})$$

$$I = \int_S(-15x^2y^2)\,d(x,y) = -30\cdot\int_0^{\pi/2}\int_0^{\sqrt[6]{\sin(2\varphi)}} a^2\rho^2\cos^2(\varphi)\cdot b^2\rho^2\sin^2(\varphi)\cdot ab\rho\,d\rho\,d\varphi =$$

chybné meze končí opravu!

$$= -30a^3b^3\int_0^{\pi/2}\int_0^{\sqrt[6]{\sin(2\varphi)}} \rho^5\cdot\cos^2(\varphi)\cdot\sin^2(\varphi)\,)\,d\rho\,d\varphi =$$

$$= -5a^3b^3\int_0^{\pi/2}\cos^2(\varphi)\cdot\sin^2(\varphi)\cdot\left[\rho^6\right]_0^{\sqrt[6]{\sin(2\varphi)}}d\varphi =$$

$$= -5a^3b^3\int_0^{\pi/2}\frac{1}{4}\sin^2(2\varphi)\cdot\sin(2\varphi)\,d\varphi = \left|\begin{aligned} u &= \cos(2\varphi) \\ du &= -2\sin(2\varphi)\,d\varphi\end{aligned}\right| =$$

$$= -\frac{5}{16}a^3b^3\int_{-1}^{1}\sqrt{1-u^2}\cdot\frac{1}{2}\,du = -\frac{5}{16}a^3b^3\int_0^1(1-u^2)\,du =$$

$$= -\frac{5}{16}a^3b^3\left[u-\frac{u^3}{3}\right]_0^1 = -\frac{5}{16}\cdot\frac{2}{3}a^3b^3 = -\frac{5}{24}a^3b^3$$

12 b

$$x^2+y^2+z^2+16u = 20^{2}\ !!$$
$$z^3-2z^2y-4xyz+8x^2y^2=0$$

$$\bar{a}=(a_x,a_y)=(1,1)$$

Analýza úlohy: $z=z(x,y) \quad u=u(x,y)$

$$\det\left(\frac{D(F,G)}{D(z,u)}\right)=\begin{vmatrix} 2z & 16 \\ 3z^2-4zy-4xy & 0 \end{vmatrix} = -16(3z^2-4zy-4xy) \neq 0 =: J$$

$$2+z^2+16u=20$$
$$z^3-2z^2-4z+8=0 \Rightarrow z(z^2-4)-2(z^2-4) \equiv (z-2)(z^2-4)=0 \Rightarrow \begin{matrix} z_1=2 \\ z_2=-2 \end{matrix}$$

$$\bar{\lambda}=(1,1,-2,1) \qquad \bar{\mu}=(1,1,2,1) \qquad \Rightarrow z=-2$$

$$J(\bar{\lambda})=-16(12+8-4)=-16^2\neq 0 \qquad J(\bar{\mu})=-16(12-8-4)=0\ !!$$

Řešení:

$$3z^2\frac{\partial z}{\partial x}-4z\frac{\partial z}{\partial x}y-4yz-4xy\frac{\partial z}{\partial x}+16xy^2=0$$

$$\frac{\partial z}{\partial x}=-\frac{-4yz+16xy^2}{3z^2-4yz-4xy} \qquad \frac{\partial z}{\partial x}(\bar{a})=-\frac{8+16}{16}=-\frac{3}{2}$$

$$\frac{\partial z}{\partial y}=-\frac{-2z^2-4xz+16x^2y}{3z^2-4yz-4xy} \qquad \frac{\partial z}{\partial y}(\bar{a})=-\frac{-8+8+16}{16}=-1$$

$$\frac{\partial^2 z}{\partial x^2}=-\frac{\left(-4y\frac{\partial z}{\partial x}+16y^2\right)(3z^2-4yz-4xy)-(-4yz+16xy^2)\left(6z\frac{\partial z}{\partial x}-4y\frac{\partial z}{\partial x}-4y\right)}{(3z^2-4yz-4xy)^2}$$

$$\frac{\partial^2 z}{\partial x^2}(\bar{a})=-\frac{\left(-4\frac{3}{2}+16\right)\cdot 16-24\cdot\left(\underbrace{12\cdot\frac{3}{2}}_{18}+\underbrace{4\frac{3}{2}-4}_{2}\right)}{16^2}=-\frac{22\cdot 16-24\cdot 20}{16^2}$$

$$\frac{\partial^2 z}{\partial x^2}(\bar{a})=\frac{1}{2}$$

$$\frac{\partial^2 z}{\partial y^2}=-\frac{\left(-4z\frac{\partial z}{\partial y}-4x\frac{\partial z}{\partial y}+16x^2\right)(3z^2-4yz-4xy)-(-2z^2-4xz+16x^2y)\left(6z\frac{\partial z}{\partial y}-4z-4y\frac{\partial z}{\partial y}-4x\right)}{(3z^2-4yz-4xy)^2}$$

$$\frac{\partial^2 z}{\partial x}(\bar{a})=-\frac{(-8+4+16)\cdot 16-16\cdot(12+8+4-4)}{16^2}=-\frac{12\cdot 16-16\cdot 20}{16^2}$$

$$\frac{\partial^2 z}{\partial y}(\bar{a})=\frac{1}{2}$$

$$\Delta z(\bar{a})=\left(\frac{1}{2}+\frac{1}{2}\right)=1$$

9 bodů

$x, y, z > 0$

❸ (bodů) Vyšetřete lokální extrémy funkce $g(x,y,z) = xy^2z(10-2x-y-z)$ na množině $(0,+\infty)^3$.

$$\left.\begin{aligned} \frac{\partial g}{\partial x} &= y^2z(10-2x-y-z) - 2xy^2z \overset{!}{=} 0 \\ \frac{\partial g}{\partial y} &= 2xyz(10-2x-y-z) - xy^2z \overset{!}{=} 0 \\ \frac{\partial g}{\partial z} &= xy^2(10-2x-y-z) - xy^2z \overset{!}{=} 0 \end{aligned}\right\} \Rightarrow \begin{aligned} -2\cancel{x}^2\cancel{y}^2\cancel{z} + \cancel{x}y^3\cancel{z} - \cancel{x}\cancel{y}^2z^2 &= 0 \\ -2x + y - z &= 0 \\ z &= y - 2x \end{aligned}$$

$$y^{\cancel{2}}\cancel{z}(10-2x-y-z) - \cancel{2xy^2z} = 4x\cancel{yz}(10-2x-y-z) - \cancel{2xy^2z}$$

$$y = 4x \quad \vee \quad 10-2x-y-z = 0 \text{ (to ale nelze!)}$$

$$\Downarrow$$

$$z = 2x \quad \Rightarrow \quad \vec{a} = (1,4,2)$$

chybně určený stacionární bod KONČÍ OPRAVU!

$$\left\{\begin{aligned} \frac{\partial^2 g}{\partial x^2} &= -2y^2z - 2y^2z = \overbrace{-4y^2z}^{-128} \\ \frac{\partial^2 g}{\partial z^2} &= -xy^2 - xy^2 = \underbrace{-2xy^2}_{-32} \\ \frac{\partial^2 g}{\partial x \partial z} &= \underbrace{y^2(10-2x-y-z) - y^2z - 2xy^2}_{32-32-32} \end{aligned}\right.$$

$$\frac{\partial^2 g}{\partial y^2} = 2xz(10-2x-y-z)\overbrace{-2xyz-2xyz}^{-4xyz}$$

$$\frac{\partial^2 g}{\partial x \partial y} = 2yz(10-2x-y-z) - y^2z - 4xyz$$

$$\frac{\partial^2 g}{\partial y \partial z} = \overbrace{2xy(10-2x-y-z)}^{16}\overbrace{-2xyz-xy^2}^{-16-16}$$

$$10-2x-y-z|_{\vec{a}} = 2$$

4
16
24
4
64

$$H_{\vec{a}} = \begin{pmatrix} -128 & -32 & -32 \\ -32 & -24 & -16 \\ -32 & -16 & -32 \end{pmatrix}$$

$$H_{\vec{a}} \cdot \frac{1}{16} = \begin{pmatrix} -8 & -2 & -2 \\ -2 & -3/2 & -1 \\ -2 & -1 & -2 \end{pmatrix}$$

$$\Delta_1 = -8 < 0$$
$$\Delta_2 = 12 - 4 = 8 > 0$$
$$\Delta_3 = -24 - 4 - 4 + 6 + 8 + 8 = -10 < 0$$

Podle Sylvesterova kritéria $d^2g_{\vec{a}}(dx, dy, dz) \equiv \vec{dx}^T H_{\vec{a}} \vec{dx} \lhd 0$

$\Rightarrow$ detekovaný stacionární bod $\vec{a} = (1,4,2)$ je bodem lokálního maxima

O definitnosti ale muselo být rozhodnuto korektně!

6b

❹ (bodů) Nechť $a, b > 0$ jsou pevně zvolené parametry. Zapište parametrizaci množiny

$$A = \left\{ (x, y) \in \mathbf{E}^2 : \frac{x^4}{a^4} + \frac{y^4}{b^4} = \frac{x^2}{a^2} \wedge x \geqslant 0 \wedge y \geqslant 0 \right\}.$$

•) Jedná se o křivky $\Rightarrow$ hledáme křivkovou parametrizaci

$$\left.\begin{aligned} x &= a\rho\sqrt{\cos\varphi} \\ y &= b\rho\sqrt{\sin\varphi} \end{aligned}\right\} \Rightarrow \frac{x^4}{a^4} + \frac{y^4}{b^4} = \frac{x^2}{a^2} \Rightarrow \rho^4 = \rho^2\cos\varphi \Rightarrow$$

$$\Rightarrow \rho = \sqrt{\cos\varphi} \Rightarrow \varphi \in (0; \pi/2)$$

$$x, y \geq 0 \Rightarrow \varphi \in (0; \pi/2)$$

Parametrizace:

$$\vec{p}(\rho; \varphi) = (a \cdot \cos\varphi; b\sqrt{\cos\varphi \cdot \sin\varphi}); \quad \varphi \in (0; \pi/2)$$

8b

❺ (bodů) Dvoudimenzionální míra $\mu(X)$ je zadána prostřednictvím dvou vytvořujících funkcí:

$$\psi(y) = y^3 \quad \& \quad \varphi(x) = \begin{cases} x & x < 0, \\ 1 + x^2 & x \geqslant 0. \end{cases} \tag{1}$$

Určete obsah elipsy

$$E = \left\{(x, y) \in \mathbf{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1\right\}.$$

$\varphi(x) \notin C^1(\mathbb{R}) \Rightarrow$ množinu E je nutno rozdělit na 3 disjunktní části, v nichž $\varphi(x)$ bude třídy C^1 $\Rightarrow$

$A = \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leqq 1 \wedge x < 0\}$

$B = \{ \text{———} \text{〃} \text{———} \wedge x = 0\} = \{0\} \times \langle -b, b \rangle$

$C = \{ \text{———} \text{〃} \text{———} \wedge x > 0\}$

$$\mu(A) = \int_A 1 \, d\mu(x,y) = \int_A 1 \cdot 3y^2 \, d(x,y) = \left| \begin{matrix} x = a\rho\cos\varphi \\ y = b\rho\sin\varphi \end{matrix} \right| = \int_{\pi/2}^{3\pi/2}\int_0^1 (3b^2\rho^2 \underbrace{\cos^2(\varphi)}_{\sin^2(\varphi)}) ab\rho \, d\rho \, d\varphi =$$

$$= ab^3 \cdot 3 \int_{\pi/2}^{3\pi/2} \left[\frac{\rho^4}{4}\right]_0^1 \sin^2(\varphi)\,d\varphi = \frac{3}{4}ab^3 \int_{\pi/2}^{3\pi/2} \cancel{\cos^2(\varphi)}\,\sin^2(\varphi)\, d\varphi = \frac{3}{8}ab^3 \int_{\pi/2}^{3\pi/2} (1 - \cos(2\varphi))\, d\varphi =$$

$$= \frac{3}{8}ab^3 \left[\varphi - \frac{1}{2}\sin(2\varphi)\right]_{\pi/2}^{3\pi/2} = \frac{3}{8}ab^3\pi$$

$$\mu(B) = \mu(\{0\}) \cdot \mu(\langle -b, b \rangle) = \mu_\varphi^{(\alpha)}(\{0\}) \cdot \mu(\langle -b, b \rangle) = \inf_{\varepsilon > 0} F(\langle 0; \varepsilon)) \cdot F(\langle -b, b \rangle) =$$

$$= \inf_{\varepsilon > 0} (1 + \varepsilon^2) \cdot (b^3 - (-b)^3) = 1 \cdot 2b^3 = 2b^3$$

$$\mu(C) = \int_C 1 \, d\mu(x,y) = \int_C 2x \cdot 3y^2 \, d(x,y) = \int_{-\pi/2}^{\pi/2} \int_0^1 6a\rho\cos\varphi \, b^2\rho^2 \sin^2(\varphi))\, ab\rho \, d\rho \, d\varphi =$$

$$= 6a^2b^3 \int_{-\pi/2}^{\pi/2} \int_0^1 \rho^4 \cos\varphi \cdot \sin^2(\varphi)\, d\rho \, d\varphi = \frac{6}{5}a^2b^3 \int_{-\pi/2}^{\pi/2} \cos\varphi \cdot \sin^2(\varphi))\, d\varphi =$$

$$= \frac{6}{5}a^2b^3 \left[\frac{\sin^3(\varphi)}{3}\right]_{-\pi/2}^{\pi/2} = \frac{2}{5}a^2b^3(1+1) = \frac{4}{5}a^2b^3$$

$$\mu(E) = \frac{3}{8}ab^3\pi + 2b^3 + \frac{4}{5}a^2b^3$$

6 bodů

6 (bodů) Pro funkci

$$g(x,y)=\begin{cases}\frac{y^2x^2}{(y^2+x^2)^{3/2}}+2x+3y+1 & \dots\ (x,y)\neq(0,0)\\ 1 & \dots\ (x,y)=(0,0)\end{cases}$$

vypočítejte směrovou derivaci ve směru $\vec{s}=(1,1)$ v bodě $(0,0)$ a hodnotu

$$\frac{\langle\vec{s}\,|\,\mathrm{grad}\,g(0,0)\rangle}{\|\vec{s}\|}$$

a diskutujte vztah mezi oběma výsledky. Jaký fakt z něho vyplývá?

$\|\vec{s}\|=\sqrt{2}$

$$\vec{a}+\tau\cdot\vec{s}=\vec{0}+\tau(1,1)=(\tau;\tau)$$

$$g(\tau,\tau)=\frac{\tau^4}{2\sqrt{2}\tau^3}+2\tau+3\tau+1=5\tau+1+\frac{\sqrt{2}}{4}\tau$$

$$\frac{\partial g}{\partial\vec{s}}(0,0)=\frac{1}{\sqrt{2}}\lim_{\tau\to0}\frac{g(\tau,\tau)-g(0,0)}{\tau}=\frac{1}{\sqrt{2}}\lim_{\tau\to0}\frac{5\tau+1-1+\frac{\sqrt{2}}{4}\tau}{\tau}=\cancel{\frac{6}{\sqrt{2}}=3\sqrt{2}}\quad \frac{5}{\sqrt{2}}+\frac{1}{4}$$

$$\frac{\partial g}{\partial x}(0,0)=\lim_{\tau\to0}\frac{g(\tau,0)-g(0,0)}{\tau}=\lim_{\tau\to0}\frac{2\tau+1-1}{\tau}=2$$

$$\frac{\partial g}{\partial y}(0,0)=\lim_{\tau\to0}\frac{g(0,\tau)-g(0,0)}{\tau}=\lim_{\tau\to0}\frac{3\tau+1-1}{\tau}=3$$

$$\mathrm{grad}\,g(0,0)=(2,3)$$

$$\frac{1}{\|\vec{s}\|}\langle\vec{s}\,|\,\mathrm{grad}\,g(0,0)\rangle=\frac{1}{\sqrt{2}}\langle(1,1)\,|\,(2,3)\rangle=\frac{5}{\sqrt{2}}$$

$$\frac{5}{\sqrt{2}}\neq\cancel{\frac{6}{\sqrt{2}}}\ \frac{5}{\sqrt{2}}+\frac{1}{4}\Rightarrow$$ v bodě $(0,0)$ nemůže mít zadaná funkce zcela jistě totální diferenciál! (podle jisté věty)

8 bodů

$f(x,y) = 2x + 6y$ $\qquad$ $M = \{(x,y) \in E^2 : x^2 + y^2 - 6x + 8y \leqq 15\}$

žádné lokální nevázané

$L(x,y) = 2x + 6y + \lambda(x^2 + y^2 - 6x + 8y - 15)$

$\frac{\partial L}{\partial x} = 2 + \lambda(2x - 6) = 0 \quad \Rightarrow \quad \lambda x = 3\lambda - 1$

$\frac{\partial L}{\partial y} = 6 + \lambda(2y + 8) = 0 \quad \Rightarrow \quad \lambda y = -4\lambda - 3$

$$\lambda^2 x^2 + \lambda^2 y^2 - 6\lambda^2 x + 8\lambda^2 y = 15\lambda^2$$

$$(3\lambda - 1)^2 + (4\lambda + 3)^2 - 6\lambda(3\lambda - 1) - 8\lambda(4\lambda + 3) = 15\lambda^2$$

$$9\lambda^2 \cancel{-6\lambda} + 1 + 16\lambda^2 + 9 + \cancel{24\lambda} - 18\lambda^2 + \cancel{6\lambda} - 32\lambda^2 \cancel{- 24\lambda} = 15\lambda^2$$

$$-40\lambda^2 + 10 = 0$$

$$\lambda^2 = \frac{1}{4}$$

$\lambda = 1/2$	$\lambda = -1/2$
$\frac{1}{2}x = \frac{3}{2} - 1 \; ; \; \frac{1}{2}y = -2 - 3$	$-\frac{1}{2}x = -\frac{3}{2} - 1 \; ; \; -\frac{1}{2}y = 2 - 3$
$\bar{a} = (1, -10)$	$\bar{b} = (5, 2)$

$f(\bar{a}) = 2 - 60 = -58$ min $\qquad$ $f(\bar{b}) = 10 + 12 = 22$ max

M... kompaktní (neprázdná) množina

$f(x,y)$... spojitá funkce na M

$\Rightarrow f(M)$ je kompaktní množina

$$f(M) = Ran(f) = \langle -58, 22 \rangle$$

10b

$$\left.\begin{aligned} x &= \rho\cos\varphi \\ y &= \rho\sin\varphi \\ z &= h \end{aligned}\right\} \Rightarrow \underbrace{\rho^4 = \rho^2(\cos^2\varphi - \sin^2\varphi)}_{\rho = \sqrt{\cos(2\varphi)}} \;\&\; h = 7$$

$$\rho = \sqrt{\cos(2\varphi)} \quad \underset{x>0}{\Rightarrow} \quad \varphi \in (\cancel{0}; \pi/4) \quad -\pi/4$$

Z je křivkou v $\mathbb{R}^3$
a její parametrizací je:

$$\vec{p}(\varphi) = (\sqrt{\cos(2\varphi)}\cdot\cos\varphi;\ \sqrt{\cos(2\varphi)}\cdot\sin\varphi;\ 7)$$

$$\langle \alpha, \beta\rangle = \langle -\pi/4; \pi/4 \rangle$$

nebo zde

$$\frac{d\vec{p}}{d\varphi} = \left(-\frac{\sin(2\varphi)}{\sqrt{\cos(2\varphi)}}\cos\varphi - \sqrt{\cos 2\varphi}\cdot\sin\varphi;\ \frac{-\sin(2\varphi)}{\sqrt{\cos(2\varphi)}}\sin\varphi + \sqrt{\cos(2\varphi)}\cos\varphi; 0\right)$$

$$= \frac{1}{\sqrt{\cos 2\varphi}}\left(-\sin(2\varphi)\cos\varphi - \cos(2\varphi)\sin\varphi;\ -\sin(2\varphi)\cdot\sin\varphi + \cos(2\varphi)\cos\varphi; 0\right)$$

$$\left\|\frac{d\vec{p}}{d\varphi}\right\|^2 = \frac{1}{\cos(2\varphi)}\left[\sin^2(2\varphi)\cos^2(\varphi) + \cos^2(2\varphi)\sin^2(\varphi) + 2\sin(2\varphi)\cos(2\varphi)\cos\varphi\sin\varphi + \right.$$
$$\left. + \sin^2(2\varphi)\sin^2(\varphi) + \cos^2(2\varphi)\cos^2(\varphi) - 2\sin(2\varphi)\cos(2\varphi)\cos\varphi\sin\varphi\right] =$$

$$= \frac{1}{\cos(2\varphi)}\left[\sin^2(2\varphi) + \cos^2(2\varphi)\right] = \frac{1}{\cos(2\varphi)}$$

$$z(x^3 - xy^2) = 7\sqrt{\cos(2\varphi)}\cos(\varphi)\cdot\left[\cos(2\varphi)\cos^2(\varphi) - \cos(2\varphi)\cdot\sin^2(\varphi)\right] =$$
$$= 7\sqrt{\cos(2\varphi)}\cdot\cos(\varphi)\cdot\cos^2(2\varphi) = 7\cos^{5/2}(2\varphi)\cdot\cos(\varphi)$$

$$I = \int_{-\pi/4}^{\pi/4} 7\cos^{5/2}(2\varphi)\cdot\cos(\varphi)\cdot\frac{1}{\sqrt{\cos(2\varphi)}}\,d\varphi = 7\int_{-\pi/4}^{\pi/4}\cos^2(2\varphi)\cdot\cos(\varphi)\,d\varphi =$$

$$= 7\int_{-\pi/4}^{\pi/4}\frac{1}{2}\left(1 - 2\sin^2(\varphi)\right)\cdot\cos(\varphi)\,d\varphi = \frac{7}{2}\int_{-\pi/4}^{\pi/4}\left(1 - 2\sin^2(\varphi)\right)\cdot\cos(\varphi)\,d\varphi =$$

$$= \left|\begin{aligned} u &= \sin(\varphi) \\ du &= \cos(\varphi)\,d\varphi \end{aligned}\right| = \frac{7}{2}\int_{-\sqrt{2}/2}^{\sqrt{2}/2}(1 - 2u^2)\,du = \frac{7}{2}\left[u - \frac{2}{3}u^3\right]_{-\sqrt{2}/2}^{\sqrt{2}/2} =$$

$$= 2\cdot\frac{7}{2}\left[u\left(1 - \frac{2}{3}u^2\right)\right]_0^{\sqrt{2}/2} = 7\cdot\frac{\sqrt{2}}{2}\left(1 - \frac{2}{3}\frac{1}{2}\right) = 7\cdot\frac{\sqrt{2}}{2}\cdot\frac{2}{3} = \frac{7}{3}\sqrt{2}$$

$$x^4 \frac{\partial^2 u}{\partial x^2} + x^2y^2 \frac{\partial^2 u}{\partial y^2} + 2x^3y \frac{\partial^2 u}{\partial x \partial y} - 2x^2 u = 0$$

$$\left.\begin{aligned} r &= \frac{y}{x} \\ s &= \frac{x^2}{y} \end{aligned}\right\} \quad \det\left(\frac{D(r,s)}{D(x,y)}\right) = \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ \frac{2x}{y} & -\frac{x^2}{y^2} \end{vmatrix} = \frac{1}{y} - 2\frac{1}{y} = -\frac{1}{y} \neq 0$$

13 B

$$M_{reg} = \{(x,y) \in \mathbb{R}^2 : x \neq 0 \wedge y \neq 0\}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\left(-\frac{y}{x^2}\right) + \frac{\partial u}{\partial s} 2\frac{x}{y} \quad \& \quad \frac{\partial u}{\partial y} = \frac{1}{x}\frac{\partial u}{\partial r} - \frac{x^2}{y^2}\frac{\partial u}{\partial s}$$

$$x^4 / \ \frac{\partial^2 u}{\partial x^2} = \frac{y^2}{x^4}\frac{\partial^2 u}{\partial r^2} - 4\frac{1}{x}\frac{\partial^2 u}{\partial r \partial s} + 4\frac{x^2}{y^2}\frac{\partial^2 u}{\partial s^2} + 2\frac{y}{x^3}\frac{\partial u}{\partial r} + 2\frac{1}{y}\frac{\partial u}{\partial s}$$

$$x^2y^2 / \ \frac{\partial^2 u}{\partial y^2} = \frac{1}{x^2}\frac{\partial^2 u}{\partial r^2} - 2\frac{x}{y^2}\frac{\partial^2 u}{\partial r \partial s} + \frac{x^4}{y^4}\frac{\partial^2 u}{\partial s^2} + 0 + 2\frac{x^2}{y^3}\frac{\partial u}{\partial s}$$

$$2x^3y / \ \frac{\partial^2 u}{\partial x \partial y} = -\frac{y}{x^3}\frac{\partial^2 u}{\partial r^2} + \left(\frac{1}{y} + 2\frac{1}{y}\right)\frac{\partial^2 u}{\partial r \partial s} - 2\frac{x^3}{y^3}\frac{\partial^2 u}{\partial s^2} - \frac{1}{x^2}\frac{\partial u}{\partial r} - 2\frac{x}{y^2}\frac{\partial u}{\partial s}$$

$$\frac{\partial^2 u}{\partial r^2}\left[y^2 + y^2 - 2y^2\right] + \frac{\partial^2 u}{\partial r \partial s}\left[-4x^3 - 2x^3 + 6x^3\right] + \frac{\partial^2 u}{\partial s^2}\left[4\frac{x^6}{y^2} + \frac{x^6}{y^2} - 4\frac{x^6}{y^2}\right] +$$

$$\frac{\partial u}{\partial r}\left[2xy - 2xy\right] + \frac{\partial u}{\partial s}\left[2\frac{x^4}{y} + 2\frac{x^4}{y} - 4\frac{x^4}{y}\right] - 2x^2 u = 0$$

$$\frac{x^6}{y^2}\frac{\partial^2 u}{\partial s^2} - 2x^2 u = 0 \Rightarrow \frac{x^4}{y^2}\frac{\partial^2 u}{\partial s^2} - 2u = 0$$

$$s^2 \frac{\partial^2 u}{\partial s^2} - 2u(r,s) = 0$$

$$x^2y'' - 2y = 0 \quad x = e^t \quad y' = \frac{1}{x}\dot{y} \ \& \ y'' = -\frac{1}{x^2}\dot{y} + \frac{1}{x^2}\ddot{y}$$

$$\underbrace{-\dot{y} + \ddot{y} - 2y = 0} \quad \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2) \stackrel{!}{=} 0 \quad \Rightarrow \underbrace{FS = \{e^{-t}; e^{2t}\}}$$

$$u(r,s) = g(r)\cdot\frac{1}{s} + h(r)\cdot s^2$$

$$u(x,y) = g\left(\frac{y}{x}\right)\cdot\frac{y}{x^2} + h\left(\frac{y}{x}\right)\cdot\frac{x^4}{y^2}$$

5b

$$\mathbb{H}_{\vec{a}} = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} \Rightarrow f(\vec{x}): \mathbb{E}^3 \mapsto \mathbb{R}, \text{ tj. } n = 3$$

$$\Rightarrow \vec{a} = (1,1,1) \qquad f(\vec{0}) = 0 \qquad \& \qquad \underbrace{\operatorname{grad} f(\vec{a}) = 0}_{\Rightarrow\, df_{\vec{a}}(dx,dy,dz) = 0}$$

✓ nápad s Taylorovým radom

$$f(x,y,z) = f(\vec{a}) + df_{\vec{a}}(dx,dy,dz) + \frac{1}{2!} d^2 f_{\vec{a}}(dx,dy,dz) =$$

$$= \alpha + 0 + \frac{1}{2!}\Big[3(x-1)^2 + 3(y-1)^2 + 3(z-1)^2 + 6(x-1)(y-1) + 6(x-1)(z-1) + 6(y-1)(z-1)\Big]$$

$$= \alpha + \frac{3}{2}(x^2+y^2+z^2) - 9(x+y+z) + 3(xy+xz+yz) + \frac{27}{2} =$$

$$= \alpha + \frac{3}{2}(x+y+z-3)^2 \quad \& \quad f(\vec{0}) \overset{!}{=} 0 \Rightarrow \alpha = -\frac{27}{2}$$

$$f(x,y,z) = -\frac{27}{2} + \frac{3}{2}(x+y+z-3)^2$$

$$\frac{x^4}{a^4} + \frac{y^4}{b^4} = \frac{x^2}{a^2} \quad \wedge \quad x \geq 0 \ \wedge \ y \geq 0$$

6b

$$\left.\begin{aligned} x &= a\rho\sqrt{\cos\varphi} \\ y &= b\rho\sqrt{\sin\varphi} \end{aligned}\right\} \Rightarrow \quad \rho^4 = \rho^2 \cos(\varphi) \quad \Rightarrow \quad \rho = \sqrt{\cos\varphi}$$

Jedná se o plochu v $\mathbb{R}^3$ $\Rightarrow$ hledáme plošnou parametrizaci

je-li prováděna křivková parametrizace nelze získat nic jak 1 bod celkem

$$\vec{p}(\varphi; h) = (a\cos(\varphi);\ b\sqrt{\cos\varphi \cdot \sin\varphi};\ h)$$

$$\underbrace{G_{\varphi, h} = \{(\varphi, h) \in \mathbb{R}^2 :\ \varphi \in (0; \pi/2) \wedge h \in \mathbb{R}\}}$$

$$T = \left\{ (x,y,z) \in \mathbb{R}^3 : \ \frac{x^4}{a^4} + \frac{y^2}{b^2} + \frac{z^4}{c^4} \leq 1 \right\}$$

Jedna verze řešení!

Označme: $U = \left\{ (x,y,z) \in \mathbb{R}^3 : \ \frac{x^4}{a^4} + \frac{y^2}{b^2} + \frac{z^4}{c^4} \leq 1 \ \wedge \ x,y,z > 0 \right\}$

tj. „$U = \frac{1}{8} T$"

1. transformace

$$\left.\begin{aligned} x^2 &= \xi \\ y &= \lambda \\ z^2 &= \eta \end{aligned}\right\} \Rightarrow \det\left(\frac{D(\xi,\eta,\lambda)}{D(x,y,z)} \right) = \begin{vmatrix} 2x & 0 & 0 \\ 0 & 2z & 0 \\ 0 & 0 & 1 \end{vmatrix} = 4xz$$

$$\Rightarrow \det\left(\frac{D(x,y,z)}{D(\xi,\eta,\lambda)} \right) = \frac{1}{4\sqrt{\xi\eta}}$$

$$\tilde{U} := \left\{ (\xi,\eta,\lambda) \in \mathbb{R}^3 : \ \frac{\xi^2}{a^4} + \frac{\eta^2}{c^4} + \frac{\lambda^2}{b^2} \leq 1 \ \wedge \ \xi,\eta,\lambda > 0 \right\}$$

2. transformace

$$\left.\begin{aligned} \xi &= a^2 \rho \cos\vartheta \cos\varphi \\ \eta &= c^2 \rho \cos\vartheta \sin\varphi \\ \lambda &= b \rho \sin\vartheta \end{aligned}\right\} \Rightarrow \det\left(\frac{D(\xi,\eta,\lambda)}{D(\rho,\vartheta,\varphi)} \right) = a^2 c^2 b \rho^2 \cos\vartheta$$

$$\tilde{\tilde{U}} = \left\{ (\rho,\vartheta,\varphi) \in \mathbb{R}^3 : \ \vartheta \in (0; \tfrac{\pi}{2}) \wedge \varphi \in (0; \tfrac{\pi}{2}) \wedge \rho \leq 1 \right\}$$

Výpočet:

$$\lambda_3(U) = \int_U 1 \, d(x,y,z) = \int_{\tilde{U}} \frac{1}{4\sqrt{\xi\eta}} \, d(\xi,\eta,\lambda) =$$

$$= \frac{1}{4} \cancel{\frac{1}{ac}} \int_{\tilde{\tilde{U}}} \cancel{\frac{1}{\rho}} \cancel{\frac{1}{\cos\vartheta}} \frac{1}{\sqrt{\cos\varphi \sin\varphi}} \, \cancel{a^2 c^2} b \cancel{\rho^2} \cancel{\cos\vartheta} \, d(\vartheta,\varphi,\rho) =$$

$$= \frac{1}{4} abc \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} \rho \frac{1}{\sqrt{\cos\varphi \sin\varphi}} \, d\vartheta \, d\varphi \, d\rho = \frac{1}{8} abc \cdot \frac{\pi}{2} \int_0^{\pi/2} \frac{1}{\sqrt{\cos\varphi \sin\varphi}} \, d\varphi =$$

$$= \frac{\pi}{16} abc \cdot \underbrace{\frac{2\Gamma(\frac{1}{4}) \cdot \Gamma(\frac{5}{4})}{\sqrt{\pi}}}_{\text{(TABULKA)}} = \frac{\sqrt{\pi}}{8} abc \, \Gamma(\tfrac{1}{4}) \cdot \Gamma(\tfrac{5}{4})$$

Pozn. $\frac{\Gamma(\frac{1}{4}) \cdot \Gamma(\frac{1}{4})}{2\Gamma(\frac{1}{2})} = \frac{2}{\sqrt{\pi}} \Gamma(\tfrac{1}{4}) \cdot \Gamma(\tfrac{5}{4})$

$\Leftarrow \Gamma(\alpha+1) = \alpha \Gamma(\alpha)$

$$\Rightarrow \lambda_3(T) = 8 \lambda_3(U) = \sqrt{\pi} \, abc \, \Gamma(\tfrac{1}{4}) \cdot \Gamma(\tfrac{5}{4})$$

Druhá verze řešení

$$T = \left\{ (x,y,z) \in \mathbb{R}^3 : \frac{x^4}{a^4} + \frac{y^2}{b^2} + \frac{z^4}{c^4} \leqq 1 \right\}$$

$$U = \frac{1}{8} T \quad \text{(viz první verze řešení)}$$

$$x = a\rho \sqrt{\cos\vartheta \cos\varphi}$$
$$y = b\rho^2 \sin\vartheta$$
$$z = c\rho \sqrt{\cos\vartheta \sin\varphi}$$

$$\det \frac{D(x,y,z)}{D(\rho,\vartheta,\varphi)} = abc \begin{vmatrix} \sqrt{\cos\vartheta\cos\varphi} & -\frac{1}{2}\frac{\sin\vartheta}{\sqrt{\cos\vartheta}}\sqrt{\cos\varphi}\cdot\rho & -\rho\sqrt{\cos\vartheta}\frac{\sin\varphi}{\sqrt{\cos\varphi}}\cdot\frac{1}{2} \\ 2\rho\sin\vartheta & \rho^2\cos\vartheta & 0 \\ \sqrt{\cos\vartheta\sin\varphi} & -\frac{1}{2}\frac{\sin\vartheta}{\sqrt{\cos\vartheta}}\sqrt{\sin\varphi}\,\rho & \rho\sqrt{\cos\vartheta}\frac{\cos\varphi}{\sqrt{\sin\varphi}}\frac{1}{2} \end{vmatrix} =$$

$$= \dots = \frac{abc\,\rho^3}{2\sqrt{\cos\varphi\cdot\sin\varphi}}$$

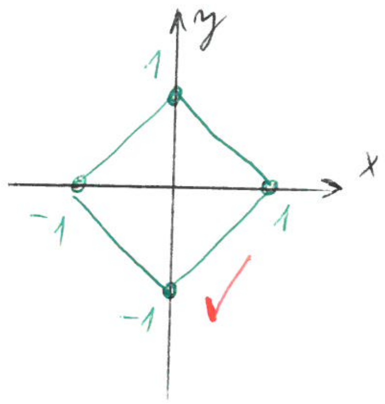
$$\lambda_3(U) = \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} \frac{1}{2} abc\,\rho^3 \frac{1}{\sqrt{\cos\varphi\cdot\sin\varphi}}\, d\varphi\, d\vartheta\, d\rho =$$

$$= \frac{1}{2} abc \cdot \frac{1}{4} \cdot \frac{\pi}{2} \int_0^{\pi/2} \frac{1}{\sqrt{\cos\varphi\sin\varphi}}\, d\varphi = \frac{\pi}{16} abc \frac{\Gamma(\frac{1}{4})\cdot\Gamma(\frac{1}{4})}{2\cdot\Gamma(\frac{1}{2})}$$

$$\lambda_3(T) = 8\cdot\lambda_3(U) = \frac{\pi}{4} abc \frac{\Gamma(\frac{1}{4})\cdot\Gamma(\frac{1}{4})}{2\Gamma(\frac{1}{2})} = \frac{\pi}{4} abc \frac{\Gamma(\frac{1}{4})\cdot\frac{1}{4}\Gamma(\frac{1}{4})}{2\cdot\frac{1}{4}\cdot\frac{\sqrt{\pi}}{2}} =$$

$$= \underline{\sqrt{\pi}\, abc\, \Gamma(\tfrac{1}{4})\cdot\Gamma(\tfrac{5}{4})}$$

Pokyn pro opravy: neurčil-li student, že se jedná o křivkový integrál 1. druhu, žádné body se neudělují!



◦) 1. Q. $\vec{\varphi}(t) = (t, 1-t)$ & $t \in \langle 0, 1 \rangle$

$\dot{\vec{\varphi}}(t) = (1, -1)$ & $\|\dot{\vec{\varphi}}\| = \sqrt{2}$

$$\int_{C_{1/4}} x^2 y^2 \, d\mu_c(x,y) = \int_0^1 t^2(1-t)^2 \cdot \sqrt{2} \, dt =$$

$$= \int_0^1 (t^2 - 2t^3 + t^4)\sqrt{2}\, dt = \sqrt{2}\left[\frac{t^3}{3} - \frac{2}{4}t^4 + \frac{t^5}{5}\right]_0^1 =$$

$$= \sqrt{2}\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{\sqrt{2}}{30}$$

$$\Rightarrow \int_C x^2 y^2 \, d\mu_c(x,y) = 4 \cdot \frac{\sqrt{2}}{30} = 2\,\frac{\sqrt{2}}{15}$$

$$32(x^2+y^2) = z^2 + 32u^3$$
$$4(x+y) = 8 + 4u - z$$
$\Rightarrow z = z(x,y)$, $u = u(x,y)$ ✓

$\vec{a} = (a_x, a_y) = (-1, 1)$

9b

•) Nutno najít generující bod: $\vec{\lambda} = (-1, 1, z_0, u_0) = ?$

$$\left.\begin{aligned} 64 &= z_0^2 + 32u_0^3 \\ 0 &= 8 + 4u_0 - z_0 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} 64 &= 64 + 64u_0 + 16u_0^2 + 32u_0^3 \\ 2u_0^3 + u_0^2 + 4u_0 &= 0 \\ u_0 (\underbrace{2u_0^2 + u_0 + 4}) &= 0 \end{aligned}$$

✓

$b^2 - 4ac = 1 - 32 < 0$ (nemá řešení)

$$\Rightarrow (u_0, z_0) = (0; 8) \Rightarrow \vec{\lambda} = (-1, 1, 8, 0) \ \& \ z(\vec{a}) = 8 \wedge u(\vec{a}) = 0$$ ✓

•) Je treba najít grad $z(\vec{a})$!

✓
$$\left\{\begin{aligned} 32 \cdot 2 \cdot x &= 2z \frac{\partial z}{\partial x} + 3 \cdot 32 \cdot u^2 \frac{\partial u}{\partial x} \\ 4 &= -4\frac{\partial u}{\partial x} - \frac{\partial z}{\partial x} \end{aligned}\right. \quad \begin{aligned} &\overset{\vec{\lambda}}{\Rightarrow} -64 = 16 \frac{\partial z}{\partial x}(\vec{a}) \\ &\Rightarrow \frac{\partial z}{\partial x} = 4 + 4\frac{\partial u}{\partial x} \end{aligned}$$

$$\frac{\partial u}{\partial x}(\vec{a}) = -2$$

$$\Rightarrow \frac{\partial z}{\partial x}(\vec{a}) = -4$$ ✓

✓
$$\left\{\begin{aligned} 32 \cdot 2 \cdot y &= 2z \frac{\partial z}{\partial y} + 3 \cdot 32 \cdot u^2 \frac{\partial u}{\partial y} \\ 4 &= 4\frac{\partial u}{\partial y} - \frac{\partial z}{\partial y} \end{aligned}\right\} \overset{\lambda}{\Rightarrow} \quad 64 = 16 \frac{\partial z}{\partial y}(\vec{a}) \Rightarrow \frac{\partial z}{\partial y}(\vec{a}) = 4$$

$$\frac{\partial u}{\partial y}(\vec{a}) = 2$$

$$\text{grad}\, z(\vec{a}) = (-4, 4)$$ ✓

•)
$$\frac{\partial z}{\partial \vec{s}}(\vec{a}) = \frac{1}{\|\vec{s}\|} \langle \text{grad}\, z(\vec{a}) \,|\, \vec{s} \rangle = \frac{1}{\|\vec{s}\|} \langle (-4,4) | \vec{s} \rangle \overset{!}{=} \underbrace{0}$$ ✗

↑ nestoupá ani neklesá

$\Rightarrow \vec{s}$ musí být kolmé na $(-4, 4)$ ✗

$\Rightarrow \vec{s} = (1, 1)$ – ten ale není jednotkový

$$\vec{s} = \left(\frac{\sqrt{2}}{2}; \frac{\sqrt{2}}{2}\right)$$ ✓

$$\frac{\partial u}{\partial \vec{s}}(\vec{a}) = \left\langle (-2, 2) \,\middle|\, \left(\frac{\sqrt{2}}{2}; \frac{\sqrt{2}}{2}\right) \right\rangle = 0$$

$\alpha = 0°$

$f(x,y,z) = xy + xz + yz$ 8b

$g(x,y,z) = 4x + 2y + z - 7 = 0$

$L(x,y,z) = f + \lambda \cdot g$

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= y + z + 4\lambda \overset{!}{=} 0 \\ \frac{\partial L}{\partial y} &= x + z + 2\lambda \overset{!}{=} 0 \\ \frac{\partial L}{\partial z} &= x + y + \lambda \overset{!}{=} 0 \end{aligned}\right\} \Rightarrow$$

stacionárním bodem je

$\vec{a} = (-1, 3, 5)$

$\lambda_{\vec{a}} = -2$

není-li stac. bod určen správně, nelze žádné další body získat!

$$d^2 L_{\vec{a}}(dx, dy, dz) = 2\,dx\,dy + 2\,dx\,dz + 2\,dy\,dz$$

Redukce:

$$4dx + 2dy + dz = 0$$

$$dz = -4dx - 2dy$$

$$d^2 L^{(red)}_{\vec{a}}(dx, dy) = 2\,dx\,dy + 2(dx+dy)(2dx+dy)\cdot(-2) =$$

$$= 2\,dx\,dy - 8dx^2 - 4dy^2 - 12\,dx\,dy =$$

$$= -4\left[dy^2 + 2dx^2 + \frac{5}{2}\,dx\,dy\right] =$$

$$= -4\left[\left(dy + \frac{5}{4}dx\right)^2 + \left(2 - \frac{25}{16}\right)dx^2\right] < 0$$

korektní určení typu definitnosti

$\Rightarrow$ v bodě $\vec{a} = (-1, 3, 5)$ nabývá funkce své maximální hodnoty (lokálně a vzhledem k rovině $4x + 2y + z - 7 = 0$)

$$H(a) = \int_0^{+\infty} e^{-x^2} \cos(ax)\, dx$$

9b

•) <u>Předpoklady věty:</u>

α) $H(0) = \int_0^{+\infty} e^{-x^2} dx = \sqrt{\int_0^{\infty} e^{-x^2} dx \cdot \int_0^{\infty} e^{-x^2} dx} \overset{F.V.}{=}$

$= \sqrt{\int_0^{+\infty}\int_0^{\infty} e^{-x^2-y^2} d(x,y)} = \left| \begin{matrix} x = r\cos\varphi \\ y = r\sin\varphi \end{matrix} \right| =$

$= \sqrt{\int_0^{\pi/2}\int_0^{+\infty} r e^{-r^2} dr\, d\varphi} = \sqrt{\frac{\pi}{2} \cdot \left[-\frac{1}{2} e^{-r^2}\right]_0^{+\infty}} = \frac{1}{2}\sqrt{\pi}$ ✓

β) integrand by měl být měřitelný, ale $e^{-x^2}\cos(ax)$ je spojitá, a tudíž měřitelná funkce ✓

γ) $\left| \frac{\partial}{\partial a}\left(e^{-x^2}\cos(ax)\right)\right| = \left| -e^{-x^2} \cdot x \cdot \sin(ax)\right| \leq x \cdot e^{-x^2}$

$\int_0^{+\infty} x e^{-x^2} dx = \left[-\frac{1}{2} e^{-x^2}\right]_0^{\infty} = \frac{1}{2} \in \mathbb{R} \Rightarrow x \cdot e^{-x^2} \in \mathcal{L}(\mathbb{R}^+)$ ✓

•) <u>Řešení:</u>

$$\frac{dH}{da} = -\int_0^{\infty} x \cdot e^{-x^2} \sin(ax)\, dx = \left| \begin{matrix} u = \sin(ax) & v' = x \cdot e^{-x^2} \\ u' = a \cdot \cos(ax) & v = -\frac{1}{2} e^{-x^2} \end{matrix} \right| =$$

$$= \left[\frac{1}{2}\sin(ax) \cdot e^{-x^2}\right]_0^{\infty} - \frac{a}{2}\int_0^{\infty} e^{-x^2}\cos(ax)\, dx = -\frac{a}{2} H(a)$$ ✓

Výsledkem je diferenciální rovnice:

$$\frac{dH}{da} = -\frac{a}{2} H(a)$$

$$H(a) = C \cdot e^{-a^2/4}$$ ✓

Kalibrace konstanty:

$$H(0) \overset{!}{=} \frac{1}{2}\sqrt{\pi} \Rightarrow C = \frac{1}{2}\sqrt{\pi}$$

$$\int_0^{\infty} e^{-x^2}\cos(ax)\, dx = \frac{\sqrt{\pi}}{2} e^{-a^2/2}$$ ✓

$$K = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leqq R^2\}$$

8b

+1 navíc!

$$\frac{d\varphi}{dx} = 1 \quad \& \quad \frac{d\psi}{dy} = 2y\,\Theta(y) \quad \& \quad \frac{d\omega}{dz} = 3z^2$$

$$\mu_3(K) = \int_K 1 \, d\mu_3(x,y,z) = \int_K 1 \cdot 2y\,\Theta(y) \cdot 3z^2 \, d(x,y,z) =$$

$$= \left| \begin{array}{l} x = \rho\cos\vartheta\cos\varphi \\ y = \rho\cos\vartheta\sin\varphi \\ z = \rho\sin\vartheta \end{array} \right| = 6 \int_0^R \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \rho\cos\vartheta\sin\varphi \cdot \rho^2 \sin^2\vartheta \cdot \underbrace{\rho^2\cos\vartheta}_{\text{Jacobi}} \, d\varphi\, d\vartheta\, d\rho =$$

mělo to být i tady

$$= 6 \cdot \int_0^R \rho^5 \, d\rho \cdot \int_{-\pi/2}^{\pi/2} \cos^2\vartheta \sin^2\vartheta \, d\vartheta \int_0^{2\pi} \sin\varphi \cdot \Theta(\rho\cdot\cos\vartheta\cdot\sin\varphi) \, d\varphi =$$

$$= \left| \begin{array}{l} \rho > 0 \\ \cos\vartheta > 0 \end{array} \Rightarrow \Theta(\ldots) = \Theta(\sin\varphi) \right| = 2R^6 \int_0^{\pi/2} \cos^2\vartheta \sin^2\vartheta \cdot \int_0^{2\pi} \sin\varphi \cdot \Theta(\sin\varphi) \, d\varphi =$$

← nějaký smysluplný postup

$$= \frac{1}{2}R^6 \int_0^{\pi/2} \sin^2(2\vartheta)\, d\vartheta \int_0^{\pi} \sin\varphi \, d\varphi = \frac{1}{2}R^6 \int_0^{\pi/2} (1-\cos(4\vartheta))\, d\vartheta \cdot [-\cos\varphi]_0^{\pi} =$$

$$= \frac{1}{2}\pi R^6$$

integrand je lichý v „x"

$$\int_K x \cdot d\mu_3(x,y,z) = \int_K x \cdot 2y\,\Theta(y)\, 3z^2 \, d(x,y,z) = 0$$

integrand je lichý v „z"

$$\int_K z \cdot d\mu_3(x,y,z) = \int_K 2y\,\Theta(y)\, 3z^3 \, d(x,y,z) = 0$$

$$\int_K y \, d\mu_3(x,y,z) = 6 \cdot \int_0^R \rho^6 \, d\rho \cdot \int_{-\pi/2}^{\pi/2} \cos^3(\vartheta)\sin^2(\vartheta)\, d\vartheta \cdot \int_0^{\pi} \sin^2(\varphi)\, d\varphi =$$

$$= 6 \cdot \frac{R^7}{7} \cdot 2 \cdot \int_0^1 (1-u^2)\cdot u^2 \, du \cdot \frac{1}{2} \int_0^{\pi} (1-\cos(2\varphi))\, d\varphi =$$

$$= \frac{18}{7}R^7 \cdot \left[\frac{u^3}{3} - \frac{u^5}{5}\right]_0^1 \cdot \frac{1}{2} \cdot \pi = \frac{9}{7}\pi R^7 \left(\frac{1}{3} - \frac{1}{5}\right) = \frac{\cancel{9}^{\,3}}{7} \cdot \frac{2}{\cancel{15}_{\,5}} \pi R^7$$

$$\Rightarrow \vec{T} = \left(0; \frac{6}{35}\pi R^7 \cdot \frac{2}{\pi R^6}; 0\right) = \left(0; \frac{18}{35}R; 0\right)$$

$$x^2\frac{\partial^2 f}{\partial x^2} - y^2\frac{\partial^2 f}{\partial y^2} = 40x^5y \quad r = xy;\ s = \frac{x^2}{y^2}$$

9 bodů

$$\begin{vmatrix} y & x \\ \frac{2x}{y^2} & -\frac{x^2}{y^3}2 \end{vmatrix} = -2\frac{x^2}{y^2} - 2\frac{x^2}{y^2} = -4\frac{x^2}{y^2} \neq 0$$

$$M_{reg} = \{(x,y)\in\mathbb{R}^2 : x\neq 0 \wedge y\neq 0\}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}y + 2\frac{x}{y^2}\frac{\partial f}{\partial s} \quad \& \quad \frac{\partial f}{\partial y} = x\frac{\partial f}{\partial x} - 2\frac{x^2}{y^3}\frac{\partial f}{\partial y}$$

$$x^2\cdot \Big/ \ \frac{\partial^2 f}{\partial x^2} = y^2\frac{\partial^2 f}{\partial r^2} + 4\frac{x}{y}\frac{\partial^2 f}{\partial r\partial s} + 4\frac{x^2}{y^4}\frac{\partial^2 f}{\partial s^2} + \frac{2}{y^2}\frac{\partial f}{\partial s}$$

$$y^2\cdot \Big/ \ \frac{\partial^2 f}{\partial y^2} = x^2\frac{\partial^2 f}{\partial r^2} - 4\frac{x^3}{y^3}\frac{\partial^2 f}{\partial r\partial s} + 4\frac{x^4}{y^6}\frac{\partial^2 f}{\partial s^2} + 6\frac{x^2}{y^4}\frac{\partial f}{\partial s}$$

$$8\frac{x^3}{y}\frac{\partial^2 f}{\partial r\partial s} + \left(2\frac{x^2}{y^2} - 6\frac{x^2}{y^2}\right)\frac{\partial f}{\partial s} = 40x^5y$$

$$8\frac{x^3}{y}\frac{\partial^2 f}{\partial r\partial s} - 4\frac{x^2}{y^2}\frac{\partial f}{\partial s} = 40x^5y \quad \Big/\cdot \frac{1}{8}\frac{y}{x^3}$$

$$\frac{\partial^2 f}{\partial r\partial s} - \frac{1}{2}\frac{1}{xy}\frac{\partial f}{\partial s} = 5x^2y^2$$

$$\frac{\partial^2 f}{\partial r\partial s} - \frac{1}{2}\frac{1}{r}\frac{\partial f}{\partial s} = 5r^2 \qquad u = \frac{\partial f}{\partial s}$$

$$\frac{du}{dr} - \frac{1}{2r}u = 5r^2 \quad \Big/ \ \text{I.F.} = e^{-\frac{1}{2}\ln r} = \frac{1}{\sqrt{r}}$$

← integrační faktor

$$\left(\frac{1}{\sqrt{r}}\cdot u\right)' = 5r^{3/2} = \left(2\cdot r^{5/2}\right)'$$

$$u = C\cdot\sqrt{r} + 2r^3$$

$$f(r,s) = C\cdot\sqrt{r}\cdot s + 2r^3\cdot s + D$$

$$f(x,y) = C\sqrt{xy}\cdot\frac{x^2}{y^2} + 2x^3y^3\cdot\frac{x^2}{y^2} + D =$$

$$= C\sqrt{xy}\,\frac{x^2}{y^2} + 2x^5y + D$$

1. $\emptyset \in \mathcal{A}$

5b

2. $\mu(\emptyset) = 0$

3. $A, B \in \mathcal{A} \wedge A \cap B = \emptyset \wedge A \cup B \in \mathcal{A} \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$

4. $A, B \in \mathcal{A} \wedge A \subset B \Rightarrow \mu(A) \leqq \mu(B)$

5. $\forall A \in \mathcal{A}: \mu(A) \geqq 0$

Pátý axiom může být dokázán z ostatních:

$$\left.\begin{array}{l} \forall A \in \mathcal{A} \text{ jistě platí: } \emptyset \subset A \\ \text{Dále víme, že } \mu(\emptyset) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} \cancel{\text{jistě}}\ \mu(\emptyset) \leqq \mu(A) \\ 0 \leqq \mu(A) \end{array}$$

$\Rightarrow$ axiomatické zavedení míry je tzv. redundantní (stačily by 4 axiomy)

Integrand $f(x) = \dfrac{e^{-ax^6} - e^{-bx^6}}{x^4}$ není v nule vůbec definovaný, tudíž $g(x) \notin C(\langle 0; +\infty))$

Ovšem platí:

$$\lim_{x\to 0_+} f(x) \overset{L'H}{=} \lim_{x\to 0_+} \frac{6x^5(b\cdot e^{-bx^6} - a e^{-ax^6})}{4x^3} = \frac{3}{2}\cdot \lim_{x\to 0_+} x^2(b\cdot e^{-bx^6} - a e^{-ax^6}) = 0$$

Tedy $f(x)$ je λ-ekvivalentní se spojitou funkcí

$$h(x) = \begin{cases} \dfrac{e^{-ax^6} - e^{-bx^6}}{x^4} & x > 0 \\ 0 & x = 0 \end{cases}$$

Naplnění předpokladů věty o derivaci integrálu podle parametru:

$$G(a,b) := \int_0^\infty f(x \mid a, b)\, dx$$

α) $G(a=b) = 0$

β) $g(x) \sim h(x) \in C(\langle 0; +\infty)) \Rightarrow g(x) \in \Lambda_\lambda(\langle 0; +\infty))$

γ) $\left| \dfrac{\partial f}{\partial a} \right| = \left| \dfrac{-x^6 e^{-ax^6}}{x^4} \right| = \underbrace{x^2 e^{-ax^6} \leq x^2 e^{-\omega x^6}}\ [0 < \omega < a]$

← nezávislost majorant na parametru!

Důkaz, že $x^2 e^{-\omega x^6} \in \mathcal{L}(\langle 0; +\infty))$:

$$\underbrace{\int_0^\infty x^2 e^{-\omega x^6} dx = \left| \begin{matrix} u = x^3 \\ du = 3x^2 dx \end{matrix} \right| = \frac{1}{3}\int_0^\infty e^{-\omega u^2} du = \frac{1}{3}\cdot\frac{1}{2}\sqrt{\frac{\pi}{\omega}} \in R}_{*}$$

Výpočet:

$$\frac{\partial G}{\partial a} = \frac{\partial}{\partial a}\int_0^\infty \frac{e^{-ax^6} - e^{-bx^6}}{x^4} dx = -\int_0^\infty x^2 e^{-ax^6} = \left| \quad * \quad \right| = -\frac{1}{6}\sqrt{\frac{\pi}{a}}$$

$$G(a,b) = -\frac{1}{3}\sqrt{\pi a} + C \quad \& \quad G(b;b) = -\frac{1}{3}\sqrt{\pi b} + C \overset{!}{=} 0$$

$$\Rightarrow C = \frac{1}{3}\sqrt{\pi b}$$

Závěr:

$$\int_0^\infty \frac{e^{-ax^6} - e^{-bx^6}}{x^4} = \frac{1}{3}\sqrt{\pi}(\sqrt{b} - \sqrt{a})$$

$$\left[\left(\frac{|x|}{a}+\frac{|y|}{b}\right)^2+\frac{z^2}{c^2}\right]^2 \leq \frac{|x|}{a}-\frac{|y|}{b}$$ **9b**

$x = a\rho\cos\vartheta\cos^2\varphi$ → nelze užít pro úhly φ mimo interval $(0;\frac{\pi}{2})$ ⇒ nutno užít symetrii (viz *)

$y = b\rho\cos\vartheta\sin^2\varphi$ $\qquad \Delta_J = 2abc\rho^2\cos\vartheta\cos\varphi\sin\varphi$ ✓

$z = c\rho\sin\vartheta$ ✓

$$B = \{(\rho,\vartheta,\varphi)\in\mathbb{R}^3 : \underbrace{\rho^4 \leq \rho\cos\vartheta(\cos^2\varphi-\sin^2\varphi)}_{\rho^3 \leq \cos\vartheta\cos(2\varphi)}\}$$ ✓

$\cos\vartheta\cdot\cos(2\varphi) \overset{!}{\geq} 0 \Rightarrow \cos\vartheta > 0$ (to je ale vždy) $\Rightarrow \cos(2\varphi) \overset{!}{\geq} 0 \Rightarrow$

$\Rightarrow \underbrace{\varphi\in(-\frac{\pi}{4};\frac{\pi}{4})\cup(\frac{3}{4}\pi;\frac{5\pi}{4})}$ ✓

jsou-li některé meze chybně, dále je NEOPRAVUJI!

$$\lambda_3(B) = 4\int_{0\,(*)}^{\pi/4\,(*)}\int_{-\pi/2}^{\pi/2}\int_0^{\sqrt[3]{\cos\vartheta\cdot\cos(2\varphi)}} 2abc\rho^2\cos\vartheta\cos\varphi\sin\varphi\, d\rho\, d\vartheta\, d\varphi =$$ ✓✓

$$= \frac{4}{3}\int_0^{\pi/4}\int_{-\pi/2}^{\pi/2}\cos\vartheta\cdot\sin(2\varphi)\cdot\cos\vartheta\cdot\cos(2\varphi)\, d\vartheta\, d\varphi \cdot abc =$$

$$= \frac{4}{3}\int_0^{\pi/4}\sin(2\varphi)\cos(2\varphi)\, d\varphi\cdot\int_{-\pi/2}^{\pi/2}\cos^2(\vartheta)\, d\vartheta \cdot abc =$$ ✓ za nějaký myšlený krok ve výpočtu

$$= \frac{4}{3}\left[\frac{1}{4}\sin^2(2\varphi)\right]_0^{\pi/4}\cdot abc\cdot 2\int_0^{\pi/2}\frac{1+\cos(2\vartheta)}{2}\, d\vartheta = \frac{1}{3}\left[\vartheta+\frac{1}{2}\sin(2\vartheta)\right]_0^{\pi/2} abc =$$

$$= \frac{\pi}{6}abc$$ ✓✓

1+8

$$\left.\begin{aligned} G(x,y,z,u) &= x^2+y^2+z^2-3xyz=0 \\ H(x,y,z,u) &= u - xy^2z^3 = 0 \end{aligned}\right\} \quad \& \quad \vec{x}=(1,1,1,1) \quad \& \quad G(\vec{x})=H(\vec{x})=0$$

– zadání vede na 2 implicitní funkce: $z=z(x,y)$ & $u=u(x,y)$ ← ✓ (musí být explicitně zapsáno!)

Kontrola nekritičnosti generujícího bodu

$$\det\left(\frac{D(G,H)}{D(z,u)}\right) = \begin{vmatrix} 2z \overset{-3xy}{} & 0 \\ -3xy^2z^2 & 1 \end{vmatrix} = 2z-3xy\Big|_{\vec{x}} = 2-3=-1 \neq 0$$ ✓

Totální diferenciál funkce $u(x,y)$

$$\left.\begin{aligned} 2x + 2z\frac{\partial z}{\partial x} - 3yz - 3xy\frac{\partial z}{\partial x} &= 0 \\ \frac{\partial u}{\partial x} - y^2z^3 - xy^2 3z^2\frac{\partial z}{\partial x} &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} 2 + 2\frac{\partial z}{\partial x}(\vec{a}) - 3 - 3\frac{\partial z}{\partial x}(\vec{a}) &= 0 \\ \frac{\partial u}{\partial x}(\vec{a}) - 1 - 3\frac{\partial z}{\partial x}(\vec{a}) &= 0 \end{aligned}$$ ✓

$$\left(\frac{\partial z}{\partial x}(\vec{a});\ \frac{\partial u}{\partial x}(\vec{a})\right) = (-1;-2)$$ ✓

$$\left.\begin{aligned} 2y + 2z\frac{\partial z}{\partial y} - 3xz - 3xy\frac{\partial z}{\partial y} &= 0 \\ \frac{\partial u}{\partial y} - 2xyz^3 - 3xy^2z^2\frac{\partial z}{\partial y} &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} 2 + 2\frac{\partial z}{\partial y}(\vec{a}) - 3 - 3\frac{\partial z}{\partial y}(\vec{a}) &= 0 \\ \frac{\partial u}{\partial y}(\vec{a}) - 2 - 3\frac{\partial z}{\partial y}(\vec{a}) &= 0 \end{aligned}$$ ✓

$$\left(\frac{\partial z}{\partial y}(\vec{a});\ \frac{\partial u}{\partial y}(\vec{a})\right) = (-1,-1)$$ ✓

TD: $du_{\vec{a}}(dx,dy) = -2dx - dy$ ✓

Rovnice tečné roviny

$$u-1 = \frac{\partial u}{\partial x}(\vec{a})\cdot(x-1) + \frac{\partial u}{\partial y}(\vec{a})\cdot(y-1)$$ ✓

$$u-1 = -2(x-1)-(y-1)$$

$$u-1 = -2x+2-y+1$$

$$\underline{2x+y+u=4}$$

extra bod za úpravu do hezkého tvaru

✓

$$\left.\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \end{aligned}\right\} \Rightarrow (x^2+y^2)^2 = a^2(x^2-y^2) \Rightarrow r = \sqrt{\cos(2\varphi)}$$

9b

Pokud někdo chybně určil typ integrálu (zde se jedná o křivkový integrál), nezískával žádný bod.

Parametrizace křivky:

$$\vec{r}(\varphi) = \left(a\sqrt{\cos(2\varphi)}\cdot\cos\varphi\,;\ a\sqrt{\cos(2\varphi)}\cdot\sin\varphi\right)$$

viz zadání

$$\text{interval: } \quad \varphi \in \langle 0; \pi/4) \quad \Leftarrow \quad \cos(2\varphi) \geq 0 \ \wedge \ \underbrace{x, y > 0}$$

Norma parametrizace:

$$\|\dot{\vec{r}}\|^2 = \dot{x}^2 + \dot{y}^2 = \frac{a^2}{\cos(2\varphi)}$$

Výpočet integrálu:

jsou-li chybné meze (typicky $\frac{\pi}{2}$ pro horní mez), dále se neopravuje

$$\int_C xy \, d\mu_C(x,y) = \int^{\pi/4} a^2 \cdot \cos(2\varphi)\cdot\cos\varphi\cdot\sin\varphi \cdot \frac{a}{\sqrt{\cos(2\varphi)}}\, d\varphi =$$

$$= \frac{1}{2}a^3 \int_0^{\pi/4} \sqrt{\cos(2\varphi)}\cdot\sin(2\varphi)\, d\varphi = \left| \begin{aligned} u &= \cos(2\varphi) \\ du &= -2\cdot\sin(2\varphi)\, d\varphi \end{aligned} \right| =$$

$$= \frac{1}{4}a^3 \int_0^1 \sqrt{u}\, du = \frac{a^3}{4}\left[\frac{2}{3}u^{3/2}\right]_0^1 = \frac{a^3}{6}$$

pouze správný numerický výsledek!

$$L(x,y,z) = yz + 2x(y+z) + \lambda(4x^2 + y^2 + z^2 - 12) \qquad x,y,z \overset{!}{>} 0$$

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= 2y + 2z + 8\lambda x \overset{!}{=} 0 \\ \frac{\partial L}{\partial y} &= z + 2x + 2\lambda y \overset{!}{=} 0 \\ \frac{\partial L}{\partial z} &= y + 2\cancel{y}^{x} + 2\lambda z \overset{!}{=} 0 \end{aligned}\right\} \Rightarrow \quad y = z \quad \& \quad \begin{aligned} 4y + 8\lambda x &= 0 \quad / \cdot y \\ y + 2x + 2\lambda y &= 0 \quad / \cdot (-1) \cdot x \cdot 4 \end{aligned}$$

$$4y^2 - 4xy - 8x^2 = 0$$

$$y^2 - xy - 2x^2 = 0$$

$$\left(y - \frac{x}{2}\right)^2 = \frac{9}{4}x^2$$

$$\left|y - \frac{x}{2}\right| = \frac{3}{2}|x| \Rightarrow y = 2x$$

Stacionární bod:

$$\vec{a} = (1,2,2); \quad \lambda = -1$$

Hessova matice

$$H = \begin{pmatrix} 8\lambda & 2 & 2 \\ 2 & 2\lambda & 1 \\ 2 & 1 & 2\lambda \end{pmatrix} \qquad H|_{\vec{a}} = \begin{pmatrix} -8 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}$$

$$d^2L_{\vec{a}}(dx,dy,dz) = -8dx^2 - 2dy^2 - 2dz^2 + 4dxdy + 4dxdz + 2dydz$$

Redukce

$$G(x,y,z) = 4x^2 + y^2 + z^2 - 12 \Rightarrow \operatorname{grad} G(x,y,z) = (8x; 2y; 2z)$$

$$\Rightarrow \operatorname{grad} G(\vec{a}) = (8; 4; 4) \Rightarrow dG_{\vec{a}}(dx,dy,dz) = 8dx + 4dy + 4dz = 0$$

$$dz = -2dx - dy$$

$$\frac{1}{2}d^2L^{(red)}_{\vec{a}}(dx,dy) = -4dx^2 - dy^2 - 4dx^2 - 4dxdy - dy^2 + 2dxdy + 2dx(-2dx - dy) + dy(-2dx - dy) =$$

$$= -12dx^2 - 3dy^2 - 6dxdy = -3\left[dy^2 + 2dxdy + 4dx^2\right] =$$

$$= \underbrace{-3\left[(dy+dx)^2 + 3dx^2\right] \lhd 0}_{}$$

(korektní rozhodnutí o typu definitnosti !)

Závěr:

Funkce nabývá v bodě $\vec{a}$ ostrého vázaného maxima

```
g := (x + y - z) ^2 + (y + 2 z) ^2 + z^2 - 1
Solve[{D[g, {x, 1}] == 0, D[g, {y, 1}] == 0, g == 0}, {x, y, z}]
{{x → -3, y → 2, z → -1}, {x → 3, y → -2, z → 1}}
Expand[g]
```

$-1 + x^2 + 2xy + 2y^2 - 2xz + 2yz + 6z^2 = F(x,y,z)$

✓ = slovní popis metody

Tato rovnice zadává (ve vhodných generujících bodech) implicitní funkci $z = z(x,y)$

Podmínka existence:

$$\frac{\partial F}{\partial z} = -2x \overset{+2y}{} + 12z \overset{!}{\neq} 0$$

My hledáme globální extrém této funkce:

Stacionární body: $\quad \operatorname{grad} z(x,y) \overset{!}{=} \vec{0}$

$$\frac{\partial F}{\partial x} = 2x + 2y - 2z \quad \& \quad \frac{\partial F}{\partial y} = 2x + 4y + 2z$$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \overset{!}{\neq} 0 \quad \& \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} \overset{!}{=} 0 \quad \Rightarrow \quad \begin{aligned} x + y - z &= 0 \\ x + 2y + z &= 0 \end{aligned}$$

✓

$$2x = -3y \Rightarrow x = -\frac{3}{2}y$$
$$z = -y/2$$

$$\left(\frac{9}{4} + 2\left(-\frac{3}{2}\right) + 2 - \frac{3}{2} - 1 + \frac{3}{2}\right) y^2 = 1$$

$$\frac{1}{4}y^2 = 1 \quad \Rightarrow \quad y = \pm 2$$

$\Rightarrow$ existují 2 stacionární body: $\vec{a} = (-3, 2)$ & $z = -1$ ✓
$\vec{b} = (3, -2)$ & $z = 1$ ✓

Globální extrémy získáme porovnáním funkčních hodnot:

$$z(\vec{a}) = -1 \quad \& \quad z(\vec{b}) = 1$$

$\Rightarrow \vec{b}$ je globálním maximem

$\Rightarrow$ Na elipsoidu má bod $(3, -2, 1)$ nejvyšší složku z. ✓

! Kontrola, zda $\vec{a}, \vec{b}$ nejsou kritickými body:

$$\frac{\partial F}{\partial z}(-3, 2, -1) = 6 \overset{+4}{} - 12 = -2 \neq 0$$
$$\frac{\partial F}{\partial z}(3, -2, 1) = -6 \overset{-4}{} + 12 = 2 \neq 0$$

✓

$$\vec{F}(x,y) = (F_1(x,y);\ F_2(x,y)) = (0;\ x^2y)$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2xy \quad \leftarrow \quad \text{Pro aplikaci Greenovy věty}$$

9b

$$x = a\rho\sqrt{\cos\varphi} \qquad \& \qquad \det\left(\frac{D(x,y)}{D(\rho,\varphi)}\right) = \frac{1}{2}ab\rho\frac{1}{\sqrt{\cos\varphi\sin\varphi}}$$
$$y = b\rho\sqrt{\sin\varphi}$$

$$(\rho^4)^{5/12} \leqq \rho\sqrt{\cos\varphi} \qquad x \geq 0,\ y \geq 0$$

$$\rho \leqq (\cos\varphi)^{3/4} \quad \Rightarrow \quad \varphi \in (0; \pi/2)$$

$$I \overset{\text{Green}}{=} \iint_A 2xy\, d(x,y) = 2\int_0^{\pi/2}\int_0^{(\cos\varphi)^{3/4}} ab\rho^2\sqrt{\cos\varphi\sin\varphi}\cdot\frac{1}{2}ab\rho\frac{1}{\sqrt{\cos\varphi\sin\varphi}}\,d\rho\,d\varphi =$$

$$= a^2b^2\int_0^{\pi/2}\int_0^{(\cos\varphi)^{3/4}} \rho^3\,d\rho\,d\varphi = \frac{1}{4}a^2b^2\int_0^{\pi/2}\cos^3(\varphi)\,d\varphi =$$

$$= \frac{1}{4}a^2b^2\int_0^{\pi/2}(1-\sin^2\varphi)\cos\varphi\,d\varphi = \left|\begin{matrix} w = \sin\varphi \\ dw = \cos\varphi\,d\varphi \end{matrix}\right| = \frac{1}{4}a^2b^2\int_0^1(1-w^2)\,dw =$$

$$= \frac{1}{4}a^2b^2\left[w - \frac{w^3}{3}\right]_0^1 = \frac{1}{4}a^2b^2\frac{2}{3} = \frac{1}{6}a^2b^2$$

$$G(a) = \int_0^\infty \frac{x\left(\operatorname{arctg}\frac{x}{a} - \operatorname{arctg}\frac{x}{b}\right)}{x^2+c^2}\,dx$$

10b

1) $a = b \Rightarrow G(a) = 0$

2) integrand $x \mapsto f(x \mid a, b, c)$ je na $\langle 0; +\infty)$ spojitý $\Rightarrow$ jedná se o měřitelnou funkci

3) $$\left|\frac{\partial f}{\partial a}\right| = \left|\frac{x}{x^2+c^2} \cdot \frac{-\frac{x}{a^2}}{1+\frac{x^2}{a^2}}\right| = \frac{x^2}{(x^2+c^2)(x^2+a^2)} \leq \frac{1}{x^2+c^2}$$

a nezávisí na „a"

$$\underbrace{\int_0^\infty \frac{1}{x^2+c^2}\,dx = \frac{1}{c^2}\int_0^\infty \frac{1}{1+(\frac{x}{c})^2}\,dx = \frac{1}{c}\left[\operatorname{arctg}\frac{x}{c}\right]_0^\infty = \frac{\pi}{2c} \in \mathbb{R}} \Rightarrow \frac{1}{x^2+c^2} \in \mathcal{L}(0;+\infty)$$

Výpočet:

$$\frac{\partial G}{\partial a} = -\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+c^2)}\,dx = -\int_0^\infty \left(\frac{1}{x^2+c^2} - \frac{a^2}{(x^2+a^2)(x^2+c^2)}\right)dx =$$

$$= -\int_0^\infty \frac{1}{x^2+c^2}\,dx + a^2\int_0^\infty \frac{1}{x^2+a^2} - \frac{1}{x^2+c^2}\,dx\ \frac{1}{c^2-a^2}\,dx = \left| \begin{matrix}\text{proto musí být}\\ c \neq a\end{matrix} \right| =$$

Parc. zlomky

$$= \left(-1 - \frac{a^2}{c^2-a^2}\right)\int_0^\infty \frac{1}{x^2+c^2}\,dx + \frac{a^2}{c^2-a^2}\int_0^\infty \frac{1}{x^2+a^2}\,dx =$$

$$= \frac{-c^2}{c^2-a^2}\int_0^\infty \frac{1}{x^2+c^2}\,dx + \frac{a^2}{c^2-a^2}\int_0^\infty \frac{1}{x^2+a^2}\,dx = \frac{-c}{c^2-a^2}\left[\operatorname{arctg}\frac{x}{c}\right]_0^\infty +$$

$$+ \frac{a}{c^2-a^2}\left[\operatorname{arctg}\frac{x}{a}\right]_0^\infty = \frac{\pi}{2}\,\frac{a-c}{c^2-a^2} = \frac{\pi}{2}\,\frac{1}{c+a}$$

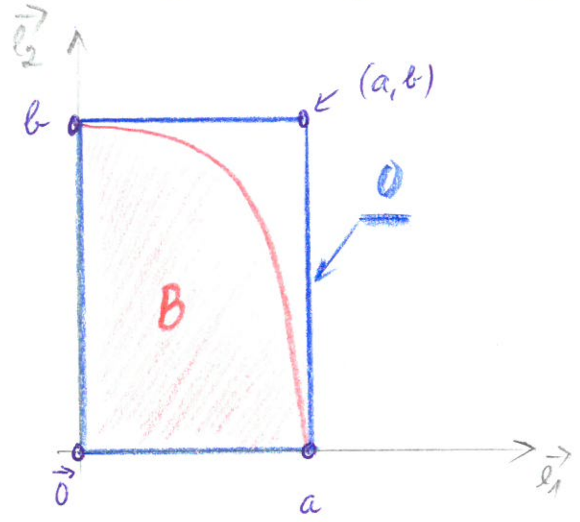
$$G(a) = \frac{\pi}{2}\ln(a+c) + \Omega \quad \& \quad G(a=b) \stackrel{!}{=} 0 = \frac{\pi}{2}\ln(b+c) + \Omega$$

$$\Rightarrow \underline{\underline{G(a,b,c) = \frac{\pi}{2}\ln\frac{a+c}{c+b}}}$$

4. $O = \langle 0,a\rangle \times \langle 0,b\rangle$

9b

$B = \left\{ (x,y)\in\mathbb{R}^2 :\ x,y\geq 0 \wedge \frac{x^3}{a^3}+\frac{y^3}{b^3}\leq 1 \right\}$



$\varphi(x) = x^3$ & $\psi(y) = y^3$

$$\mu_2(O) = \left| \begin{matrix}\text{vytvořující funkce}\\ \text{jsou spojité}\end{matrix} \right| = \mu_2(\underbrace{\langle 0,a) \times \langle 0,b)}_{\in \mathcal{X}_2}) = F_2(\langle 0,a)\times\langle 0,b)) =$$

$$= [\psi(b)-\psi(0)]\cdot[\varphi(a)-\varphi(0)] = a^3b^3$$

$$\mu_2(B) = \int_B 1 \, d\mu_2(x,y) = \int_B \frac{d\varphi}{dx}\cdot\frac{d\psi}{dy}\, d(x,y) = \int_B 3x^2\cdot 3y^2 \, d(x,y) =$$

$$= 9\int_B x^2y^2 \, d(x,y) = \left| \begin{matrix} x = \rho\cos^{2/3}\varphi \cdot a \\ y = \rho\sin^{2/3}\varphi\cdot b \end{matrix} \quad J = ab\,\frac{2}{3}\,\rho\,(\cos\varphi\sin\varphi)^{-1/3} \right| =$$

$$= 9\int_0^1\int_0^{\pi/2} a^2b^2\rho^4\cos^{4/3}\varphi\,\sin^{4/3}\varphi\cdot\frac{2}{3}ab\rho\,(\cos\varphi\sin\varphi)^{-1/3}\,d\varphi\,d\rho =$$

$$= 6a^3b^3\int_0^1\int_0^{\pi/2}\rho^5\cdot\cos\varphi\cdot\sin\varphi\,d\varphi\,d\rho = \left|\begin{matrix}\text{věta o}\\ \text{separabilitě}\end{matrix}\right| = 6a^3b^3\int_0^1\rho^5\,d\rho\cdot\int_0^{\pi/2}\cos\varphi\cdot\sin\varphi\,d\varphi =$$

$$= a^3b^3\left[\frac{\sin^2(\varphi)}{2}\right]_0^{\pi/2} = \frac{1}{2}a^3b^3$$

Poměr ploch: 50%

$$f(x,y) = \begin{cases} \dfrac{3(y+2)^2 x^4}{(y+2)^4 + 5x^8} & (x,y) \neq (0,-2) \\ 4/7 & (x,y) = (0,2) \end{cases}$$

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$$M_{\alpha,\beta} = \{(x,y) \in \mathbb{R}^2 : y = \alpha x^2 + \beta\}$$

$$(0,-2) \overset{!}{\in} M_{\alpha,\beta} \Rightarrow -2 = \alpha \cdot 0 + \beta \Rightarrow \beta \overset{!}{=} -2$$

$$\lim_{\substack{(x,y)\to(0,-2) \\ (x,y)\in M_{\alpha,\beta}}} f(x,y) = \lim_{x\to 0} \frac{3(\alpha x^2+2-2)^2 \cdot x^4}{[\alpha x^2-2+2]^4+5x^8} = \lim_{x\to 0} \frac{3\alpha^2 x^8}{\alpha^4 x^8 + 5x^8} = \frac{3\alpha^2}{\alpha^4+5} \overset{!}{=} \frac{4}{7}$$

symbolika

$$\Rightarrow \quad 21\alpha^2 = 4\alpha^4 + 20$$

$$4\alpha^4 - 21\alpha^2 + 20 = 0$$

$$\alpha^4 - \frac{21}{4}\alpha^2 + 5 = 0$$

$$(\alpha^2 - 4)(\alpha - \frac{5}{4}) = 0$$

$\Rightarrow$ čtyři řešení:

$$y = 2x - 2$$
$$y = -2x - 2$$
$$y = \frac{\sqrt{5}}{2}x - 2$$
$$y = \frac{\sqrt{5}}{2}x - 2$$

$$\left.\begin{aligned}\xi &= y + \ln x\\ \eta &= x\end{aligned}\right\} \Rightarrow \quad \det\left(\frac{D(\xi,\eta)}{D(x,y)}\right) - \begin{vmatrix}\frac{1}{x} & 1\\ 1 & 0\end{vmatrix} = \frac{1}{x} \overset{!}{\neq} 0$$

9b

·) ale $\ln x$ musí být dobře definován, proto

$$M_{\check{r}e\check{s}} = \{(x,y)\in\mathbb{R}^2 : x>0\}$$

($x \neq 0$ chybně!)

$$\frac{\partial u}{\partial x} = \frac{1}{x}\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \quad \& \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{x^2}\frac{\partial^2 u}{\partial \xi^2} + \frac{2}{x}\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} - \frac{1}{x^2}\frac{\partial u}{\partial \xi}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial \xi^2}\frac{1}{x} + \frac{\partial^2 u}{\partial \xi \partial \eta}$$

Dosazení:

$$\eta^2 \frac{\partial^2 u}{\partial \eta^2} - \eta \frac{\partial u}{\partial \eta} - 3u = 0$$

Eulerova

$$\eta = e^t \quad t = \ln \eta \quad \frac{du}{d\eta} = \frac{1}{\eta}\frac{du}{dt} \quad \frac{d^2u}{d\eta^2} = -\frac{1}{\eta^2}\frac{du}{dt} + \frac{1}{\eta^2}\frac{d^2u}{dt^2}$$

$$\ddot{u} - 2\dot{u} - 3u = 0$$

$$\lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3) \qquad FS = \{e^{-t};\ e^{3t}\}$$

$$u(\xi,\eta) = C(\xi)\cdot\frac{1}{\eta} + D(\xi)\cdot\eta^3$$

nebo za toto

$$u(x,y) = C(xe^y)\cdot\frac{1}{x} + D(xe^y)\cdot x^3$$

nebo $y + \ln x$

$$\int_0^\infty \frac{x}{x^2+1}\left(\operatorname{arctg} x - \operatorname{arctg}\frac{x}{a}\right) dx = H(a)$$ 10b

$$\frac{dH}{da} = -\int_0^\infty \frac{x}{x^2+1}\,\frac{-\frac{x}{a^2}}{1+\frac{x^2}{a^2}}\,dx = \int_0^\infty \frac{x^2}{x^2+1}\,\frac{1}{x^2+a^2}\,dx = \int_0^\infty \frac{Ax+B}{x^2+1}\,dx +$$ tvar parciálních zlomků

$$+\int_0^\infty \frac{Cx+D}{x^2+a^2}\,dx = \left| \begin{array}{l} (Ax+B)(x^2+a^2)+(Cx+D)(x^2+1) \overset{!}{=} x^2 \\ Ax^3+Aa^2x+Bx^2+Ba^2+Cx^3+Cx+Dx^2+D \overset{!}{=} x^2 \end{array}\right| =$$

$$= \left| (A,C)=\vec{0} \quad \left.\begin{array}{l} B+D=1 \\ Ba^2+D=0 \end{array}\right\} \Rightarrow \underbrace{\begin{array}{l} Ba^2+1-B=0 \\ B=\frac{1}{1-a^2} \ \& \ D=-\frac{a^2}{1-a^2}=\frac{a^2}{a^2-1} \end{array}}\right| =$$

$$= \frac{1}{1-a^2}\int_0^\infty \frac{1}{1+x^2}\,dx + \frac{a^2}{a^2-1}\int_0^\infty \frac{1}{a^2}\,\frac{1}{1+\left(\frac{x}{a}\right)^2}\,dx = \frac{1}{1-a^2}\left[\operatorname{arctg} x\right]_0^\infty +$$

$$+\frac{1}{a^2-1}\left[a\cdot\operatorname{arctg}\frac{x}{a}\right]_0^{+\infty} = \frac{1}{1-a^2}\cdot\frac{\pi}{2} + \frac{a}{a^2-1}\cdot\frac{\pi}{2} =$$

$$= \frac{\pi}{2}\cdot\frac{1}{a^2-1}(a-1) = \frac{\pi}{2}\,\frac{1}{a+1}$$

$$H(a) = \frac{\pi}{2}\ln(a+1) + C \ \wedge \ H(a=1) \overset{!}{=} 0 \Rightarrow \frac{\pi}{2}\ln 2 + C = 0$$

$$C = -\frac{\pi}{2}\ln 2$$

$$I \triangleq H(a) = \frac{\pi}{2}\ln\frac{a+1}{2}$$

Ověření předpokladů: $H(1)=0$

$$\left|\frac{x^2}{x^2+1}\cdot\frac{1}{x^2+a^2}\right| \leq \frac{1}{x^2+1} \ \wedge$$

$$\wedge \int_0^\infty \frac{1}{x^2+1}\,dx = \left[\operatorname{arctg} x\right]_0^\infty = \frac{\pi}{2} \in \mathbb{R}$$

9b

$$x = a\rho\cos\vartheta\cos^2\varphi$$
$$y = b\rho\cos\vartheta\sin^2\varphi$$
$$z = c\rho\sin\vartheta$$

$$(\rho^2\cos^2\vartheta + \rho^2\sin^2\vartheta)^{7/2} \leqq \rho^4\cos^4\vartheta - \rho^4\sin^4\vartheta$$
$$\rho^7 \leqq \rho^4(\cos^2\vartheta + \sin^2\vartheta)(\cos^2\vartheta - \sin^2\vartheta)$$
$$\rho^3 \leqq \cos(2\vartheta)$$
$$\Rightarrow \vartheta \in \left(-\frac{\pi}{4}; \frac{\pi}{4}\right) \wedge \varphi \in (0, 2\pi)$$

$$x, y, z \overset{!}{>} 0 \Rightarrow \vartheta \in (0; \pi/4) \wedge \varphi \in (0; \pi/2)$$

$$J = 2abc\rho^2\cos\vartheta\cos\varphi\sin\varphi$$

$$I = \int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt[3]{\cos 2\vartheta}} 2abc\rho^2\cos\vartheta\cos\varphi\sin\varphi \, d\rho\, d\vartheta\, d\varphi =$$

jsou-li chybné meze, dále se neopravuje!

$$= \frac{2}{3}abc \int_0^{\pi/2}\int_0^{\pi/4} \cos(2\vartheta)\cdot\cos\vartheta\cos\varphi\sin\varphi \, d\vartheta\, d\varphi = \left| \text{věta o separabilitě} \right| =$$

separace

$$= \frac{2}{3}abc \int_0^{\pi/2} \frac{1}{2}\sin(2\varphi)\, d\varphi \cdot \int_0^{\pi/4} (1 - 2\sin^2\vartheta)\cos\vartheta \, d\vartheta = \left| \begin{matrix} u = \sin\vartheta \\ du = \cos\vartheta\, d\vartheta \end{matrix} \right| =$$

$$= \frac{1}{3}abc \left[-\frac{1}{2}\cos(2\varphi)\right]_0^{\pi/2} \cdot \int_0^{\sqrt{2}/2} (1 - u^2\cdot 2)\, du =$$

$$= \frac{1}{3}abc \left[u - \frac{2}{3}u^3\right]_0^{\frac{\sqrt{2}}{2}} = \frac{1}{3}abc\cdot\frac{\sqrt{2}}{2}\left(1 - \frac{2}{3}\cdot\frac{2}{4}\right) =$$

$$= \frac{\sqrt{2}}{6}abc\left(1 - \frac{1}{3}\right) = \frac{\sqrt{2}}{6}\cdot\frac{2}{3}abc = \frac{\sqrt{2}}{9}abc$$

5b

$M_k = \{(x,y) \in \mathbb{R}^2 : y = k\cdot(x+2)\}$ $\quad (-2,0) \overset{!}{\in} M_k$

$$\lim_{\substack{(x,y)\to(-2,0)\\(x,y)\in M_k}} h(x,y) = \lim_{x\to -2}\left[2(x+5) + \frac{k(x+2)^2}{k^2(x+2)^2+(x+2)^2}\right] = 6 + \frac{k}{k^2+1}$$

symbolika!

Pro různé množiny M_k; tj. pro různá k, vycházejí limity vzhledem k množině různé $\Rightarrow$ funkce mít limitu v $(-2,0)$ nemůže

$$L = xy(z^2-9) + \lambda(3x^2+3y^2-z^4)$$

$$\frac{\partial L}{\partial x} = y(z^2-9) + 6x\lambda \stackrel{!}{=} 0$$

$$\frac{\partial L}{\partial y} = x(z^2-9) + 6y\lambda \stackrel{!}{=} 0$$

$$\frac{\partial L}{\partial z} = 2xyz - 4\lambda z^3 \stackrel{!}{=} 0$$

podmínce $x,y,z>0$ vyhovuje pouze jediný stacionární bod: $\vec{a} = (\sqrt{6};\sqrt{6};\sqrt{6})$ & $\lambda_{\vec{a}} = \frac{1}{2}$

$$\frac{\partial^2 L}{\partial x^2} = 6\lambda \quad \frac{\partial^2 L}{\partial y^2} = 6\lambda \quad \frac{\partial^2 L}{\partial z^2} = -12\lambda z^2 + 2xy \quad \frac{\partial^2 L}{\partial x \partial y} = z^2-9 = \frac{\partial^2 L}{\partial y \partial x}$$

$-36+12$

$$\frac{\partial^2 L}{\partial x \partial z} = 2yz = \frac{\partial^2 L}{\partial z \partial x} \qquad \frac{\partial^2 L}{\partial y \partial z} = 2xz = \frac{\partial^2 L}{\partial z \partial y}$$

$$d^2L_{\vec{a}}(dx,dy,dz) = 3dx^2 + 3dy^2 - 24dz^2 - 6dxdy + 24dxdz + 24dydz$$

$$d^2H_{\vec{a}}(dx,dy,dz) = dx^2 + dy^2 - 8dz^2 - 2dxdy + 8dxdz + 8dydz$$

Redukce: $g(x,y,z) = 3x^2+3y^2-z^4$

$$\frac{\partial g}{\partial x} = 6x \;\&\; \frac{\partial g}{\partial y} = 6y \;\&\; \frac{\partial g}{\partial z} = -4z^3 \Rightarrow dg_{\vec{a}}(dx,dy,dz) = 6\sqrt{6}(dx+dy) - 24\sqrt{6}dz$$

$$dx + dy - 4dz = 0 \Rightarrow dz = \frac{1}{4}(dx+dy)$$

$$d^2H^{(red)}_{\vec{a}}(dx,dy) = dx^2 + dy^2 - \frac{1}{2}dx^2 - \frac{1}{2}dy^2 - 1dxdy - 2dxdy +$$

$$+ 2dx^2 + 2dxdy + 2dxdy + 2dy^2 =$$

$$= \frac{5}{2}dx^2 + \frac{5}{2}dy^2 + dxdy$$

$$d^2\tilde{H}^{(red)}_{\vec{a}}(dx,dy) = dx^2 + dy^2 + \frac{2}{5}dxdy = \left(dx + \frac{1}{5}dy\right)^2 + \frac{4}{5}dy^2$$

$$\Rightarrow d^2H^{(red)}_{\vec{a}}(dx,dy) > 0 \Rightarrow$$

$\Rightarrow$ bod $\vec{a}$ je $OLVM_{in}$

$1+6=7$

$$\frac{x^3}{a^3} + \frac{y^3}{b^3} + \frac{z^3}{c^3} = 1 \qquad \bar{\lambda} = (x_0; y_0; z_0)$$

$$z = z(x, y) \qquad \frac{\partial F}{\partial z} = -3\frac{z^2}{c^3} \neq 0 \quad \Rightarrow \quad z \neq 0$$

$$\frac{\partial F}{\partial x} = 3\frac{x^2}{a^3} \qquad \frac{\partial F}{\partial y} = 3\frac{y^2}{b^3}$$

$$\bar{a} = (x_0; y_0) \qquad z(\bar{a}) = z_0$$

$$\frac{\partial z}{\partial x}(\bar{a}) = \frac{c^3}{a^3}\frac{x_0^2}{z_0^2} \qquad \frac{\partial z}{\partial y}(\bar{a}) = \frac{c^3}{b^3}\frac{y_0^2}{z_0^2}$$

Tečná rovina:

$$z - z_0 = \frac{c^3}{a^3}\frac{x_0^2}{z_0^2}(x - x_0) + \frac{c^3}{b^3}\frac{y_0^2}{z_0^2}(y - y_0) \qquad \Big/ \; z_0^2 \frac{1}{c^3}$$

$$\frac{z z_0^2 - z_0^3}{c^3} = \frac{x x_0^2 - x_0^3}{a^3} + \frac{y y_0^2 - y_0^3}{b^3}$$

$$\frac{x x_0^2}{a^3} + \frac{y y_0^2}{b^3} - \frac{z z_0^2}{c^3} = \frac{x_0^3}{a^3} + \frac{y_0^3}{b^3} - \frac{z_0^3}{c^3}$$

$$\frac{x x_0^2}{a^3} + \frac{y y_0^2}{b^3} + \frac{z z_0^2}{c^3} = 1$$

$\nearrow$

Zdá se, že výsledek platí pouze, pokud $z \neq 0$. Pokud by ale $z = 0$, pak by buď $x \neq 0$ nebo $y \neq 0$ a bylo by možno rozklad přes implicitní funkci $x = x(y, z)$ nebo $y = y(x, z)$

$\Downarrow$

výsledek je platný univerzálně!

$$\left.\begin{array}{l} x^2+z^2=4az \\ x^2+(z-2a)^2=4a^2 \end{array}\right\} \Rightarrow \text{parametrizace:} \quad \begin{array}{l} x = 2a\cos(\tau) \\ y = a \\ z = 2a + 2a\sin(\tau) \end{array}$$

gf

$\& \ \tau \in \langle 0; 2\pi \rangle$

$$x^2+z^2 = 4a^2\cos^2(\tau) + 4a^2 + 8a^2\sin(\tau) + 4a^2\sin^2(\tau) = 8a^2 + 8a^2\sin(\tau) = 4az \quad \boxed{\checkmark}$$

$$\vec{\varphi}(\tau) = (2a\cos(\tau);\ a;\ 2a+2a\sin(\tau))$$

$$\dot{\vec{\varphi}}(\tau) = (-2a\sin(\tau);\ 0;\ 2a\cos(\tau))$$

$$\|\dot{\vec{\varphi}}(\tau)\|^2 = 4a^2\sin^2(\tau) + 4a^2\cos^2(\tau) = 4a^2$$

$$\|\dot{\vec{\varphi}}(\tau)\| = 2a$$

$$\int_W z^2 y \, d\mu_c(x,y,z) = \int_0^{2\pi} (2a+2a\sin(\tau))^2 \cdot a \cdot 2a \, d\tau =$$

$$= 8a^4\int_0^{2\pi}(1+\sin\tau)^2 d\tau = 8a^4\int_0^{2\pi} d\tau + \underbrace{16a^4\int_0^{2\pi}\sin(\tau)\,d\tau}_{=0} + \int_0^{2\pi} 8a^4\sin^2(\tau)\,d\tau =$$

$$= 16\pi a^4 + 0 + 32a^4\int_0^{\pi/2}\frac{1-\cos(2\tau)}{2}\,d\tau = 16\pi a^4 + 16a^4\left[\tau - \frac{1}{2}\sin(2\tau)\right]_0^{\pi/2} =$$

$$= 16\pi a^4 + 8\pi a^4 = \cancel{32}\pi a^4$$

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Jedná se o kružnici ležící na válci $\underbrace{x^2+z^2=4az}_{(x-2a)^2+z^2=4a^2}$ $\}$

Katedra matematiky Fakulty jaderné a fyzikálně inženýrské ČVUT v Praze								CELKEM
Příjmení a jméno	1	2	3	4	5	6	BONUS	

Zkoušková písemná práce č. 8 z předmětu 01MAB4

27/08/2019, 9:00-11:00

1 (4 body) Vyšetřete lokální extrémy funkce $f(x,y,z) = xy + xz + yz$.

2 (9 bodů) Jaký je podíl plochy (v procentech) útvaru

$$B = \left\{(x,y) \in \mathbf{E}^2 : \frac{x^3}{a^3} + \frac{y^3}{b^3} \leqslant 1\right\}$$

ku ploše obdélníku $O = \langle -a, a\rangle \times \langle -b, b\rangle$ při volbě vytvořující funkce $\varphi(\tau) = \tau^5|\tau|$? Neopomeňte teoretické aspekty řešení (např. splnění předpokladů u použitých metod) .

3 (7 bodů)
Prověřte, zda má funkce

$$f(x,y) = \begin{cases} 2x + \frac{2x^5}{x^4+y^2} + 3y - 8 & \dots \quad (x,y) \neq (0,0), \\ -8 & \dots \quad (x,y) = (0,0), \end{cases}$$

totální diferenciál v bodě $(0,0)$. Své tvrzení podrobně komentujte!

4 (12 bodů) Nalezněte Taylorův polynom prvního stupně funkce $u = u(x,y,z)$ v bodě $\vec{a} = (x_0, y_0, z_0) = (2, 3, -1)$. Funkce $u = u(x,y,z)$ nechť je zadána rovnicí

$$u^3 - xu^2 - 4zu + y - 4u = 3.$$

Dále vypočtěte $\frac{\partial^2 u}{\partial x \partial z}(\vec{a})$.

5 (9 bodů)
Nechť $a > 0$ je zvoleno pevně. Křivku

$$W = \{(x,y,z) \in \mathbf{E}^3 : x^2 + 6ax + y^2 - 8ay = 0 \wedge z = a\}$$

korektně parametrizujte a poté vypočtěte integrál

$$\int_W \frac{x^2}{z^2}\, \mathrm{d}\mu_c(x,y,z).$$

Jakou geometrickou představu o křivce máte?

6 (9 bodů)
Vypočítejte integrál $\int_U \sqrt{\frac{x}{a} + \frac{y}{b} + \frac{z}{c}}\, \mathrm{d}(x,y,z)$, kde

$$U = \left\{(x,y,z) \in \mathbf{R}^3 : x, y, z \geqslant 0 \wedge \left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right)^2 \leqslant \frac{x}{a} - \frac{y}{b}\right\}.$$

4 body

$f(x,y,z) = xy + xz + yz$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= y + z \overset{!}{=} 0 \\ \frac{\partial f}{\partial y} &= x + z \overset{!}{=} 0 \\ \frac{\partial f}{\partial z} &= x + y \overset{!}{=} 0 \end{aligned}\right\} \Rightarrow$$ jediný stacionární bod: $\vec{a} = (0,0,0)$

$$\mathbb{H} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$ $\leftarrow$ je stejná pro všechny body v $\mathbb{R}^3$, tj. platí i pro stacionární bod

$$\Rightarrow \quad d^2 f_{\vec{a}}(dx, dy, dz) = 2\,dx\,dy + 2\,dx\,dz + 2\,dy\,dz$$

Snadno lze nahlédnout, že q-forma II. TD je indefinitní, neboť např.

$$dx, dy = 1 \wedge dz = 0 \Rightarrow d^2 f_{\vec{a}} = 2$$
$$dx = 1, dz = -1, dy = 0 \Rightarrow d^2 f_{\vec{a}} = -2$$

$\Rightarrow$ Funkce $f(x,y,z)$ lokální extrémy nemá. Má jen sedlový bod.

2. $O = \langle -a,a\rangle \times \langle -b,b\rangle$ 9 bodů (+1 navíc)

$$\varphi(\tau) = \begin{cases} \tau^6 & \tau \geq 0 \\ -\tau^6 & \tau < 0 \end{cases} \rightarrow \dot{\varphi}(\tau) = \begin{cases} 6\tau^5 \\ -6\tau^5 \end{cases} = 6|\tau|^5$$

$\dot{\varphi}(\tau) \in C(\mathbb{R}) \Rightarrow \varphi(\tau) \in C^1(\mathbb{R}) \Rightarrow$ lze užít větu o převodu abstraktního integrálu na klasický (♡) ✓ teorie [bonus bod]

a)

$$\mu(A) = \underbrace{\left| \varphi(\tau) \in C(\mathbb{R}) \right|}_{✓} = \mu(\langle -a,a\rangle \times \langle -b,b\rangle) =$$

$$= (\varphi(a) - \varphi(-a)) \cdot (\varphi(b) - \varphi(-b)) = (b^6 + b^6) \cdot (a^6 + a^6) = 4a^6b^6 \checkmark$$

b)

$$\mu(B) = \int_B 1 \, d\mu(x,y) = \underbrace{\left| \begin{matrix} \text{viz} \\ (♡) \end{matrix} \right| = \int_B \frac{d\varphi}{dx} \cdot \frac{d\psi}{dy} \, d(x,y)}_{✓} =$$

$$= 36 \int_B |xy|^5 \, d(x,y) \checkmark = \left| \begin{matrix} \text{označme} \\ \tilde{B} = B \cap (\mathbb{R}^+ \times \mathbb{R}^+) \end{matrix} \right| =$$

$$= 4 \cdot 36 \int_{\tilde{B}} (xy)^5 \, d(x,y) = \left| \begin{matrix} x = ar\cos^{2/3}\varphi \\ y = br\sin^{2/3}\varphi \end{matrix} \checkmark \quad \Delta = \frac{2}{3} ab r \cos^{-1/3}\varphi \cdot \sin^{-1/3}\varphi \right| =$$

$$= 4 \cdot 36 \int_0^1 \int_0^{\pi/2} a^6 b^6 r^{10} \cos^{10/3}\varphi \cdot \sin^{10/3}\varphi \cdot \frac{2}{3} r \cos^{-1/3}\varphi \cdot \sin^{-1/3}\varphi \, d\varphi \, dr = \checkmark$$

$$= 8 \cdot 12 \int_0^1 \int_0^{\pi/2} a^6 b^6 r^{11} \cos^3(\varphi) \cdot \sin^3(\varphi) \, d\varphi \, dr = \int_0^{\pi/2} a^6 b^6 \sin^3(2\varphi) \, d\varphi \checkmark =$$

$$= \left| \begin{matrix} u = \cos(2\varphi) \\ du = -2\sin(2\varphi) \, d\varphi \end{matrix} \right| = \frac{1}{2} \int_{-1}^{1} (1 - u^2) \, du \Big|^{a^6b^6} = \int_0^1 (1 - u^2) \, du \Big|^{a^6b^6} =$$

$$= \left[u - \frac{u^3}{3} \right]_0^1 \Big|^{a^6b^6} = 1 - \frac{1}{3} \Big|_{a^6b^6} = \frac{2}{3} a^6 b^6 \checkmark$$

Podíl: $\dfrac{\frac{2}{3}a^6b^6}{4a^6b^6} = \dfrac{2}{12} = \dfrac{1}{6} = 16{,}\overline{6}\,\%$ ✓

$$u(x,y) = \begin{cases} 2x + \dfrac{2x^5}{x^4+y^2} + 3y - 8 & (x,y) \neq (0,0) \\ -8 & (x,y) = (0,0) \end{cases}$$

7b

$$\frac{\partial u}{\partial x}(0,0) = \lim_{h\to 0} \frac{1}{h}\left[2h + \frac{2h^5}{h^4} - 8 - (-8)\right] = \lim_{h\to 0} \frac{2h+2h}{h} = 4$$

$$\frac{\partial u}{\partial y}(0,0) = \lim_{h\to 0} \frac{1}{h}\left[0 + 0 + 3h - 8 - (-8)\right] = \lim_{h\to 0} \frac{3h}{h} = 3$$

$$\Rightarrow \operatorname{grad} u(0,0) = (4,3)$$

Zbytková funkce:

$$\eta(x,y) = f(x,y) - f(0,0) - 4(x-0) - 3(y-0) = 2x + \frac{2x^5}{x^4+y^2} + 3y - 8 + 8 - 4x - 3y =$$

$$= \frac{2x^5}{x^4+y^2} - 2x = \frac{-2xy^2}{x^4+y^2}$$

$$\lim_{(x,y)\to\vec{0}} \frac{\eta(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to\vec{0}} \frac{-2xy^2}{(x^4+y^2)\sqrt{x^2+y^2}}$$ ... ta ale neexistuje, neboť limity vzhledem k přímkám $y = \omega x$ jsou různé (*)

$$\lim_{\substack{(x,y)\to\vec{0} \\ (x,y)\in M_\omega}} \frac{-2xy^2}{(x^4+y^2)\sqrt{x^2+y^2}} = \left| M_\omega = \{(x,y)\in\mathbb{R}^2 : y = \omega\cdot x\} \right| =$$

$$= \lim_{x\to 0} \frac{-2x\cdot\omega^2x^2}{(x^4+\omega^2x^2)\sqrt{x^2+\omega^2x^2}} = -\lim_{x\to 0} \frac{2\omega^2}{(x^2+\omega^2)\sqrt{1+\omega^2}} = -\frac{2}{\sqrt{1+\omega^2}}$$

$$\Downarrow$$

TD v $(0,0)$ neexistuje

(plus (*))

nutno zdůvodnit!

B2 | – hledejme generující bod $\vec{\lambda} = (\lambda_x, \lambda_y, \lambda_z, \lambda_u) = (2, 3, -1, \lambda_u = ?)$ 12 bodů

$$u^3 - 2u^2 + 4u + 3 - 4u = 3$$

$$u^2(u-2) = 0 \quad \Rightarrow \text{ existují 2 generující body}$$

$$\vec{\lambda} = (2, 3, -1, 0) \quad \& \quad \vec{\mu} = (2, 3, -1, 2)$$

•) Nejsou některé z generujících bodů kritickými body?

$$\frac{\partial G}{\partial u} = 3u^2 - 2xu - 4z - 4 \qquad \frac{\partial G}{\partial u}(\vec{\lambda}) = 0 \qquad \frac{\partial G}{\partial u}(\vec{\mu}) = 12 - 8 + 4 - 4 = 4 \neq 0$$

$\Rightarrow \vec{\lambda}$ je kritickým bodem $\Rightarrow$ v řešení s ním nepokračujeme

$\Rightarrow \vec{\mu}$ není kritický $\Rightarrow$ úloha má (naštěstí !!) jen jediné řešení

•) Taylorův polynom 1. stupně

$$Tu(x,y,z) = u(\vec{a}) + \frac{\partial u}{\partial x}(\vec{a})\cdot(x-2) + \frac{\partial u}{\partial y}(\vec{a})\cdot(y-3) + \frac{\partial u}{\partial z}(\vec{a})\cdot(z+1)$$

$\Rightarrow$ nutno vyčíslit chybějící parciální derivace

$$\left.\frac{\partial G}{\partial x}\right|_{\vec{\mu}} = -u^2\Big|_{\vec{\mu}} = -4 \qquad \left.\frac{\partial G}{\partial y}\right|_{\vec{\mu}} = 1 \qquad \left.\frac{\partial G}{\partial z}\right|_{\vec{\mu}} = -4u\Big|_{\vec{\mu}} = -8$$ obecné výpočty derivací

$$\frac{\partial u}{\partial x} = \frac{4}{4} = 1 \quad \& \quad \frac{\partial u}{\partial y} = -\frac{1}{4} \quad \& \quad \frac{\partial u}{\partial z} = \frac{8}{4} = 2$$ určené hodnoty derivací

$$Tu(x,y,z) = 2 + x - 2 - \frac{1}{4}(y-3) + 2(z+1)$$

•) Výpočty vyšších derivací

$$3u^2\frac{\partial u}{\partial x} - u^2 - 2xu\frac{\partial u}{\partial x} - 4z\frac{\partial u}{\partial x} - 4\frac{\partial u}{\partial x} = 0 \quad \Big/ \frac{\partial}{\partial x}$$

$$6u\left(\frac{\partial u}{\partial x}\right)^2 + 3u^2\frac{\partial^2 u}{\partial x^2} - 2u\frac{\partial u}{\partial x} - 2u\frac{\partial u}{\partial x} - 2x\left(\frac{\partial u}{\partial x}\right)^2 - 2xu\frac{\partial^2 u}{\partial x^2} - 4z\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial x^2} = 0 \quad \Big/ \text{ dosazení}$$

$$12 + 4\cdot\frac{\partial^2 u}{\partial x^2} - 4 - 4 - 4 = 0$$

$$\frac{\partial^2 u}{\partial x^2}(\vec{a}) = 0$$

není v zadání

$\left[\frac{\partial G}{\partial x}\right]$

$$3u^2\frac{\partial u}{\partial x} - u^2 - 2xu\frac{\partial u}{\partial x} - 4z\frac{\partial u}{\partial x} - 4\frac{\partial u}{\partial x} = 0 \quad \Big/ \frac{\partial}{\partial z}$$

$\left[\frac{\partial^2 G}{\partial x \partial z}\right]$

$$6u\frac{\partial u}{\partial z}\frac{\partial u}{\partial x} + 3u^2\frac{\partial^2 u}{\partial x\partial z} - 2u\frac{\partial u}{\partial z} - 2x\frac{\partial u}{\partial z}\frac{\partial u}{\partial x} - 2xu\frac{\partial^2 u}{\partial x\partial z} - 4\frac{\partial u}{\partial x} - 4z\frac{\partial^2 u}{\partial x\partial z} - 4\frac{\partial^2 u}{\partial x\partial z} = 0$$

$$24 + 4\cdot\frac{\partial^2 u}{\partial x\partial z}(\vec{a}) - 8 - 8 - 4 = 0$$ dosazení

$$\frac{\partial^2 u}{\partial x\partial z}(\vec{a}) = -\frac{4}{4} = -1$$

! zcela správný numerický výsledek

$x^2+6ax+y^2-8ay=0$ ✓ úprava rovnice

9 ~~8~~ bodů

$$\underbrace{(x+3a)^2+(y-4a)^2=9a^2+16a^2=25a^2 \;\wedge\; z=a}$$

kružnice ležící v rovině $z=a$ na válci $(x+3a)^2+(y-4a)^2=25a^2$

✓ slovní vyjádření o tvaru objektu

Výchozí popis: $x=-3a+\rho\cos t$; $y=4a+\rho\sin t$; $z=h$

Po dosazení: $\rho^2=25a^2$ & $h=a$

odtud vyjde křivková parametrizace:

$$\vec{\varphi}(t)=(-3a+5a\cos t;\ 4a+5a\sin t;\ a) \checkmark \quad \& \quad t\in\langle 0;2\pi) \checkmark$$

$$\dot{\vec{\varphi}}(t)=(-5a\sin t;\ 5a\cos t;\ 0)$$

$$\|\dot{\vec{\varphi}}\|^2=25a^2 \Rightarrow \|\dot{\vec{\varphi}}\|=5a \checkmark$$

$$\int_W \left(\frac{x}{z}\right)^2 d\mu_1(x,y,z)=\int_0^{2\pi}(-3+5\cos t)^2\cdot 5a\,dt \checkmark =$$

$$=\int_0^{2\pi}(9-30\cos t+25\cos^2 t)\cdot 5a\,dt=5a\int_0^{2\pi}(9+25\cos^2 t)\,dt=$$

$$=5a\int_0^{2\pi}\left(9+\frac{25}{2}(1+\cos(2t))\right)dt=5a\cdot\frac{1}{2}\int_0^{2\pi}(18+25)\,dt=\boxed{\checkmark}$$

nějaký smysluplný krok

$$=5a\cdot\frac{1}{2}\cdot 2\pi\cdot 43=215a\pi \checkmark$$

②

Zvolánková Ilona

9 bodů

$$\int_U \sqrt{\frac{x}{a}+\frac{y}{b}+\frac{z}{c}}\, d(x,y,z)$$

$$U=\left\{(x,y,z)\in\mathbb{R}^3:\ x,y,z\geq 0\ \wedge\ \left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)^2\leq\frac{x}{a}-\frac{y}{b}\right\}$$

$$\left.\begin{aligned} x&=a\rho\cos^2\vartheta\cos^2\varphi\\ y&=b\rho\cos^2\vartheta\sin^2\varphi\\ z&=c\rho\sin^2\vartheta\end{aligned}\right\}\ \Delta_J=4abc\rho^2\cos^3\vartheta\sin\vartheta\cos\varphi\sin\varphi$$

$\vartheta\in(0,\frac{\pi}{2})$ $\quad\varphi\in(0,\frac{\pi}{2})$

$\rho^2\leq\rho\cos^2\vartheta\cos2\varphi$ $\qquad\rho>0$

$\rho\leq\cos^2\vartheta\cos2\varphi$ $\qquad 0=\cos^2\vartheta\cos2\varphi$

$\cos2\varphi=0\ \Rightarrow\ \varphi\in(0,\frac{\pi}{4})$

$$\int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\cos^2\vartheta\cos2\varphi}\sqrt{\rho}\,4abc\rho^2\cos^3\vartheta\sin\vartheta\cos\varphi\sin\varphi\,d\rho\,d\vartheta\,d\varphi=$$

je-li mez pro φ nebo ρ určena chybně, dále se neopravuje!

$$4abc\int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\cos^2\vartheta\cos2\varphi}\rho^{5/2}\cos^3\vartheta\sin\vartheta\cos\varphi\sin\varphi\,d\rho\,d\vartheta\,d\varphi=4abc\int\int\left[\frac{\rho^{7/2}}{\frac{7}{2}}\right]_0^{\cos^2\vartheta\cos2\varphi}\cos^3\vartheta\sin\vartheta\cos\varphi\sin\varphi\,d\vartheta\,d\varphi=$$

$$=\frac{abc}{7}\int_0^{\pi/4}\int_0^{\pi/2}\cos^{10}\vartheta\,(\cos2\varphi)^{7/2}\sin\vartheta\sin2\varphi\,d\vartheta\,d\varphi=\left|\begin{aligned}\cos2\varphi&=t\\ -2\sin2\varphi\,d\varphi&=dt\\ \cos\vartheta&=u\\ -\sin\vartheta&=du\end{aligned}\right|=$$

$$=\frac{abc}{2\cdot7}\int_{\cancel{1}\,0}^{1}\int_{\cancel{1}\,0}^{1}u^{10}t^{7/2}\,du\,dt=\frac{abc}{14}\left[\frac{u^{11}}{11}\right]_0^1\left[t^{9/2}\cdot\frac{2}{9}\right]_0^1=$$

$$=\frac{abc}{7\cdot99}=\frac{abc}{7\cdot99}=\frac{abc}{693}$$