$$\hat{L}\mathcal{E}(x,y) = \frac{\partial \mathcal{E}}{\partial t} - a^2 \frac{\partial^2 \mathcal{E}}{\partial x^2} + b \frac{\partial \mathcal{E}}{\partial x} = \delta(x,t) = \delta(x) \otimes \delta(t) \quad / \ \mathcal{F}_x$$

$$\frac{\partial E}{\partial t} - a^2(-i\xi)^2 E + b(-i\xi)E = \delta(t)$$

$$\frac{\partial E}{\partial t} + \underbrace{(a^2\xi^2 - ib\xi)}_{\omega} E = \delta(t)$$

$$\hat{K} = \frac{d}{dt} + \omega \qquad \hat{K}\mathcal{E}(t) + \omega\,\mathcal{E}(t) = \delta(t) \quad / \ \mathcal{F}_t \ (\text{nebo } \mathcal{L}_t)$$

$$-i\xi E + \omega E = 1$$

$$E = \frac{1}{\omega - i\xi} \Rightarrow \mathcal{E}(t) = \Theta(t)\, e^{-\omega t}$$

$$\Downarrow$$

$$E(\xi,t) = \Theta(t)\, e^{-(a^2\xi^2 - ib\xi)t} = \Theta(t)\, e^{-a^2\xi^2 t} \cdot e^{ib\xi t}$$

$$\mathcal{F}^{-1} = \frac{1}{2\pi}\mathcal{F}^*$$

√ metoda

$$\mathcal{E}(x,t) = \mathcal{F}^{-1}[\Theta(t)\, e^{-a^2\xi^2 t} e^{ib\xi t}] = \frac{1}{2\pi}\mathcal{F}[\Theta(t)\, e^{-a^2\xi^2 t} e^{-ib\xi t}] =$$

$$= \frac{\Theta(t)}{2\pi}\mathcal{F}[e^{-a^2\xi^2 t}](x - bt) = \frac{\Theta(t)}{2\pi}\sqrt{\frac{\pi}{a^2 t}}\, e^{-\frac{(x-bt)^2}{4a^2 t}} =$$

$$= \frac{\Theta(t)}{2a}\frac{1}{\sqrt{\pi t}}\, e^{-\frac{(x-bt)^2}{4a^2 t}}$$

√ ... za dumenzy zcela presný výsledek

9 bodů

$$\left(\lim_{n\to\infty} \theta(x,y)\, n^4 (x^2+y^2)\, e^{-n^2(x^2+y^2)} ; \varphi(x,y) \right) = \lim_{n\to\infty} \left(\theta(x,y)\, n^4 (x^2+y^2)\, e^{-n^2(x^2+y^2)} ; \varphi(x,y) \right) =$$

$$= \lim_{n\to\infty} \int_{\mathbb{R}^2} \theta(x,y)\, n^4 (x^2+y^2)\, e^{-n^2(x^2+y^2)} \varphi(x,y)\, d(x,y) =$$

$$= \lim_{n\to\infty} \int_0^{+\infty}\!\!\int_0^{+\infty} n^4 (x^2+y^2)\, e^{-n^2(x^2+y^2)} \varphi(x,y)\, d(x,y) = \left| \begin{matrix} r = nx \ \& \ s = ny \\ d(x,y) = \frac{1}{n^2} d(r,s) \end{matrix} \right| =$$

$$= \lim_{n\to\infty} \int_0^{+\infty}\!\!\int_0^{\infty} (r^2+s^2)\, e^{-r^2-s^2} \varphi\left(\tfrac{r}{n}; \tfrac{s}{n}\right) d(r,s) = \left| \text{korektní zdůvodnění záměny} \right| =$$

$$= \int_0^{\infty}\!\!\int_0^{\infty} (r^2+s^2)\, e^{-r^2-s^2} \varphi(0,0)\, d(r,s) = \left| \begin{matrix} r = \rho\cos\varphi \ \& \ s = \rho\sin\varphi \\ d(r,s) = \rho\, d(\rho,\varphi) \end{matrix} \right| =$$

$$= \int_0^{\pi/2}\!\!\int_0^{+\infty} \rho^2 e^{-\rho^2} \varphi(0,0)\, \rho\, d\rho\, d\varphi = \left| \text{věta o separabilitě} \right| = \frac{\pi}{2} \varphi(0,0) \int_0^{\infty} \rho^3 e^{-\rho^2} d\rho =$$

$$= \left| \begin{matrix} u = \rho^2 \\ du = 2\rho\, d\rho \end{matrix} \right| = \frac{\pi}{2} \varphi(0,0) \int_0^{\infty} u \cdot e^{-u} du \cdot \frac{1}{2} = \frac{\pi}{4} \varphi(0,0)$$

Závěr:

$$\underbrace{\lim_{n\to\infty} \theta(x,y)\, n^4 (x^2+y^2)\, e^{-n^2(x^2+y^2)} = \frac{\pi}{4} \delta(x) \otimes \delta(y)}$$

9 bodů

$$\int_0^\infty \frac{\sin^2(\beta x)\cdot\cos^2(\beta x)}{x^2}\,dx = \frac{1}{4}\int_0^\infty \frac{\sin^2(2\beta x)}{x^2}\,dx$$

$$\mathcal{L}[\sin^2(2\beta x)] = \mathcal{L}[1-\cos(4\beta x)]\cdot\frac{1}{2} = \frac{1}{2p} - \frac{1}{2}\,\frac{p}{p^2+16\beta^2}$$

$$\mathcal{L}[x\cdot\Theta(x)] = \frac{1}{p^2} \quad\Rightarrow\quad \mathcal{L}^{-1}\left[\frac{1}{p^2}\right] = \Theta(x)\cdot x$$

$$\Rightarrow\ \frac{1}{4}\int_0^\infty \frac{\sin^2(2\beta x)}{x}\,dx = \frac{1}{4}\int_0^\infty x\left(\frac{1}{2x} - \frac{1}{2}\,\frac{x}{p^2+16\beta^2}\right)dx = \frac{1}{8}\int_0^\infty \left(1-\frac{x^2}{x^2+16\beta^2}\right)dx =$$

$$= \frac{1}{8}\int_0^\infty \frac{x^2+16\beta^2-x^2}{x^2+16\beta^2}\,dx = 2\int_0^\infty \frac{\beta^2}{16\beta^2\left[1+\left(\frac{x}{4\beta}\right)^2\right]}\,dx = \frac{1}{8}\int_0^\infty \frac{1}{1+\left(\frac{x}{4\beta}\right)^2}\,dx =$$

$$= \frac{1}{8}\,4\beta\left[\operatorname{arctg}\frac{x}{4\beta}\right]_0^{+\infty} = \frac{\beta}{2}\cdot\frac{\pi}{2}\cdot\operatorname{sgn}(\beta) = \frac{\pi}{4}|\beta|$$

✓ ← pokud je výsledek ošetřen i pro záporné β !

(1) 2 body

$$\left(\widetilde{\lim_{k\to\infty}\mathcal{F}[f_k]}\,;\,\varphi(\xi)\right) = \lim_{k\to\infty}\left(\widetilde{\mathcal{F}[f_k]}\,;\,\varphi(\xi)\right) = \lim_{k\to\infty}\left(\tilde{f}_k\,;\,\mathcal{F}[\varphi]\right) =$$

$$= \left(\tilde{f}\,;\,\mathcal{F}[\varphi]\right) = \left(\widetilde{\mathcal{F}[f]}\,;\,\varphi(\xi)\right)$$ ✓

✓ ← vysvětlení, co který krok znamená (odkud se vzal!)

(2) 4 body ✓

$$\left(\widetilde{\lim_{k\to\infty} D^{\alpha}\mathcal{F}[f_k]}\,;\,\varphi(\xi)\right) = \lim_{k\to\infty}\left(\widetilde{D^{\alpha}\mathcal{F}[f_k]}\,;\,\varphi(\xi)\right) = (-1)^{|\alpha|}\lim_{k\to\infty}\left(\widetilde{\mathcal{F}[f_k]}\,;\,D^{\alpha}\varphi(\xi)\right) =$$

$$= \left|\text{viz bod } (1)\right| = (-1)^{|\alpha|}\left(\widetilde{\mathcal{F}[f]}\,;\,D^{\alpha}\varphi(\xi)\right) = \left(\widetilde{D^{\alpha}\mathcal{F}[f]}\,;\,\varphi(\xi)\right)$$

9b

$$y'' + y - 10\int_0^x y(\xi)\,d\xi = x - 5x^2 - 3 \qquad y(0) = 0 \;\&\; y'(0) = -4$$

$\mathcal{L}[y] =: Y(s)$

$\mathcal{L}[y'] = sY - y(0) = sY$

$\mathcal{L}[y''] = s\mathcal{L}[y'] - y'(0) = s^2 Y + 4$

$\mathcal{L}[\int_0^x y(\xi)\,d\xi] = \frac{Y}{s}$

$$\underbrace{s^2 Y + 4 + Y - 10\frac{Y}{s}} = \underbrace{-\frac{3}{s} - \frac{10}{s^3} + \frac{1}{s^2}}$$

$$Y = \frac{s - 10 - 3s^2 - 4s^3}{s^2(s^3 + s - 10)} \qquad (s^3 + s - 10) : (s - 2) = s^2 + 2s + 5$$

$$Y(s) = \frac{A}{s-2} + \frac{B}{s^2} + \frac{C}{s} + \frac{Ds + E}{s^2 + 2s + 5}$$

$$(A, B, C, D, E) = (-1, 1, 0, 1, -1)$$

$$Y(s) = \frac{-1}{s-2} + \frac{1}{s^2} + \frac{s-1}{s^2+2s+5} = -\frac{1}{s-2} + \frac{1}{s^2} + \frac{s+1}{(s+1)^2+4} - \frac{2}{(s+1)^2+4}$$

$$y(x) = \theta(x)\cdot\left[x - e^{2x} + e^{-x}\cos(2x) - e^{-x}\sin(2x)\right]$$