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# Determination of interaction potentials in freeway traffic from steady-state statistics

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## Abstract

Many-particle simulations of vehicle interactions have been quite successful in the qualitative reproduction of observed traffic patterns. However, the assumed interactions could not be measured, as human interactions are hard to quantify compared to interactions in physical and chemical systems. We show that progress can be made by generalizing a method from equilibrium statistical physics we learned from random matrix theory. It allows one to determine the interaction potential via distributions of the netto distances  $s$  of vehicles. Assuming power-law interactions, we find that driver behavior can be approximated by a forwardly directed  $1/s$  potential in congested traffic, while interactions in free traffic are characterized by an exponent of  $\alpha \approx 4$ . This is relevant for traffic simulations and the assessment of telematic systems.

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In recent years, statistical physics and nonlinear dynamics have been very successful in modeling, simulating, and understanding empirically observed instability and pattern

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formation phenomena in freeway traffic [1–6]. On the one hand, vehicle traffic flow can be well approximated as a particular one-dimensional particle gas with Boltzmann statistics [6,2]. On the other hand, the motion of a vehicle  $i$  of mass  $m$  and length  $l_i$  at location  $r_i(t)$  with speed  $v_i(t) = dr_i/dt$ , maximum velocity  $v_0$ , adaptation time  $\tau$ , and fluctuation force  $\xi_i(t)$  can be reflected by an acceleration equation of the form

$$m \frac{dv_i}{dt} = m \frac{v_0 - v_i}{\tau} + f(s_i) + \xi_i(t), \quad (1)$$

where  $f(s_i)$  is a negative and repulsive force, which depends on the netto distance (clearance)  $s_i = (r_{i+1} - r_i - l_i)$  between two successive vehicles. With  $W_i(s) = v_0 + \tau f(s)/m$ , Eq. (1) corresponds to the optimal velocity model  $dv_i/dt = [W_i(s_i) - v_i]/\tau$  [7] with an additional noise term  $\xi_i$ . Due to the lack of good single-vehicle data and suitable evaluation methods, the form of interaction potential  $U(s)$  with  $f(s) = -dU(s)/ds$  has been subject to speculation. A direct evaluation by averaging  $v_i$ -over- $s_i$  data suffers from the wide distribution of netto distances and causes problems of interpretation [2,8]. Exploiting the source of this problem, we suggest a *statistical* approach to determine the potential and present first results regarding its form, based on a comparison of single-vehicle data with the netto distance distributions obtained from rigorous solutions for a particle gas in equilibrium. For a justification of this approach see Ref. [9].

Normally, the interaction forces or potentials between particles governing each other's motion, are not directly measurable. The great success of scattering theory was to determine interaction potentials from statistical distributions of particles scattered at some "target" composed of the material under investigation [10]. Learning about human interactions requires a somewhat different approach from statistical physics, which can be learned from books on random matrix theory [11]. This method was successfully applied to the statistical description of the time gaps  $T_i$  between the arrival times of buses in some Mexican cities [12], where the potential  $U$  was found to be a logarithmic function, i.e.,  $U(T_i) = -\ln(T_i)$ . It was speculated that the same potential would approximate the (time) headway distribution of highway traffic [13]. In contrast, our study will identify the shape of the spatial interaction potential and reveal a different driver's behavior in free and congested traffic.

For this, recently extended a method developed for classical many-particle systems exposed to a "thermal bath" of a given temperature, i.e., to random forces of a certain variance and statistics [9]. The resulting velocity and netto distance distributions [see Eqs. (6) and (7)] allow one to draw conclusions about the interaction potential  $U$ , as this determines their shapes. These distributions have been originally derived assuming a conservation of momentum and energy, i.e., a transformation of potential energy into kinetic energy. However, by investigating a Fokker–Planck equation equivalent to Eq. (1), one can show that the same distributions are steady-state solutions for driven many-particle systems with forwardly directed rather than symmetrical potentials [9], if the system is large enough and if the average velocity  $V$  and variance  $\theta$  are constant. For  $dV/dt \approx 0$ ,  $d\theta/dt \approx 0$ , and  $N \gg 1$ , fluctuations in the system become negligible, and the total energy is only slightly fluctuating. The effects of the driving force  $mv_0/\tau$  and the dissipative force  $-mv_i/\tau$  balance each other in a statistical sense [9].

However,  $dV/dt \approx 0$  requires linear stability of system (1) of coupled differential equations, i.e.,  $dW(s)/ds < 1/(2\tau)$  [7]. Therefore, we have to exclude the stop-and-go regime between 20 and 40 vehicles per kilometer (veh./km) and lane from our investigation [2]. In summary, the data analysis must be carried out for ensembles of vehicles in a stationary traffic state with a well-defined velocity variance (generalized “temperature”)  $\text{Var}(v_i) = (\sigma V)^2$  in a reference frame moving with a constant velocity, the average-vehicle velocity  $V$ . That is, one has to restrict to small density intervals, otherwise one will mix up systems of different generalized temperatures and different average velocities. Such a careful analysis is presented here for the first time for single-vehicle data of the Dutch two-lane freeway A9, which was confirmed by another analysis for Czech highway data (not shown). This has led to new and surprising insights regarding the shape of the interaction potential and its dependence on the traffic state. We will focus on forwardly directed power-law potentials

$$U(s) \propto \begin{cases} s^{-\alpha} & \text{for } s > 0, \\ 0 & \text{otherwise,} \end{cases} \quad (2)$$

where  $\alpha > 0$  is a fit parameter and  $s$  the netto distance (clearance) between two successive cars on a (ring) road of length  $L$ . Similar relations have been suggested in some car-following models [14,15] and describe the repulsive tendency of drivers to keep a safe distance to the respective car ahead. The exponent determines the characteristic dynamic and stationary behavior.

*Short-ranged Dyson’s gas with power-law potential:* We will now investigate the statistical gas of  $N$  point-like identical particles on a ring of scaled length  $N$  interacting via the potential energy

$$\mathcal{U}(x_1, \dots, x_N) = C \sum_{i=1}^N U(x_{i+1} - x_i). \quad (3)$$

Herein,  $(x_1, \dots, x_N)$  is the vector of scaled particle positions  $x_i = (r_i - \sum_{j=1}^{i-1} l_j)N/L = \rho(r_i - \sum_{j=1}^{i-1} l_j)$ , where  $\rho$  denotes the global vehicle density. For convenience, we will use a periodic index, i.e.,  $x_{i+N} = x_i + N$ . Let this so-called short-ranged power-law Dyson’s gas (SRDG) be exposed to a temperature reservoir with Boltzmann statistics and generalized temperature  $\sigma^2 > 0$ . The corresponding dimensionless Hamiltonian reads

$$\mathcal{H} = \frac{1}{2} \sum_{i=1}^N (u_i - \langle u \rangle)^2 + \frac{C}{mV^2} \sum_{i=1}^N U(x_{i+1} - x_i) \quad (4)$$

with the additional condition  $\sum_{i=1}^N |x_{i+1} - x_i| = N$ , where  $u_i = v_i \langle u \rangle / V$  is the scaled velocity of the  $i$ th particle and  $\langle u \rangle = 1$  the scaled average velocity. Note that all the quantities used here are dimensionless. The probability of finding the system in the

phase-space element  $\Omega \equiv (x_1, \dots, x_N, u_1, \dots, u_N)$  is

$$P_*(\Omega) = \mathcal{N} e^{-\mathcal{H}/(2\sigma^2)} = \mathcal{N} \prod_{i=1}^N e^{-(u_i - \langle u \rangle)^2 / (2\sigma^2)} \times \prod_{i=1}^N e^{-\beta U(x_{i+1} - x_i)} \delta \left( N - \sum_{i=1}^N |x_{i+1} - x_i| \right), \tag{5}$$

where  $\beta = C / (m\sigma^2 V^2)$  represents the scaled generalized inverse temperature and  $\mathcal{N}$  the normalization factor, which can be determined from the normalization condition  $\int_{\mathbb{R}^{2N}} P_*(\Omega) d\Omega = 1$ . By integration over the  $2N - 1$  independent variables, the probability of finding the  $i$ th particle with velocity  $u_i$  is obtained as

$$P'(u_i) = \frac{1}{\sqrt{2\pi\sigma}} \exp \left[ -\frac{(u_i - \langle u \rangle)^2}{2\sigma^2} \right]. \tag{6}$$

That is, the velocities are Gaussian-distributed with mean value  $\langle u \rangle \equiv \int_{\mathbb{R}} u P'(u) du = 1$  and variance  $\langle (u - \langle u \rangle)^2 \rangle \equiv \int_{\mathbb{R}} (u - \langle u \rangle)^2 P'(u) du = \sigma^2 = \langle (v - V)^2 \rangle / V^2$ .

For the above SRDG, one can also determine the equilibrium distribution of scaled particle distances  $z_i = (x_{i+1} - x_i) = \rho s_i$  [16], which gives us the expression for the netto distance (or clearance) distributions:

$$P_\beta^{(\alpha)}(z) = A e^{-\beta z^{-\alpha}} e^{-Bz}. \tag{7}$$

Herein,  $A = A(\alpha, \beta)$  and  $B = B(\alpha, \beta)$  are constants determined via the normalization conditions

$$\int_0^\infty P_\beta^{(\alpha)}(z) dz = 1, \quad \langle z \rangle \equiv \int_0^\infty z P_\beta^{(\alpha)}(z) dz = 1. \tag{8}$$

These equations can be analytically solved only for particular potentials, including power-law relations. In the following, we need two special variants of the SRDG. In the case  $\alpha = 1$ , we find  $\int_0^\infty P_\beta^{(1)}(z) dz = 2A(\sqrt{\beta/B}) \mathcal{K}_1(2\sqrt{\beta B})$  and  $\int_0^\infty z P_\beta^{(1)}(z) dz = 2A(\beta/B) \mathcal{K}_2(2\sqrt{\beta B})$ , where  $\mathcal{K}_\lambda$  is the Mac-Donald’s function (modified Bessel’s function of the second kind) of order  $\lambda \in \mathbb{R}$ . Based on these equations, one can exactly determine the normalization constants  $A$  and  $B$ . For the case  $\alpha = 4$ , the numerically determined values of the normalization constants  $A$  and  $B$  are displayed in Table 1. For  $\beta > 2$ , we could find the approximate relations  $A \approx \exp(\alpha\beta - 0.1490\alpha^2 + 1.3689\alpha + 0.2271)$  and  $B \approx \alpha\beta + 0.4593\alpha + 0.9481$ .

*Data analysis:* We have separately analyzed eight small density intervals in the free low-density regime  $\leq 20$  veh./km and eight density intervals in the congested traffic regime  $\geq 40$  veh./km. After determination of the respective values of  $\beta$  and  $\sigma$  from the single-vehicle data, we have obtained the fit parameter  $\alpha$  by a least-square method, i.e., minimization of the error function  $\chi^2$  measuring the deviation between the theoretical and empirical clearance distributions. The predicted Gaussian distributions reproduce the empirical velocity distributions very well (see Table 2 and Fig. 1), which is also supported by other empirical and numerical studies [6]. The best fit of the netto distance

Table 1

Numerically determined values of the normalization constants  $A$  and  $B$  for  $\alpha = 4$ 

$\beta$	$A$	$B$
0	1	1
0.00001	1.158	1.0761
0.00005	1.246	1.1123
0.0001	1.3036	1.136
0.0005	1.5122	1.2151
0.001	1.6558	1.2644
0.005	2.2673	1.4405
0.01	2.783	1.5593
0.05	6.1623	2.0447
0.1	11.1941	2.4296

The values depend on  $\beta$ , and therefore, also on the density  $\rho$  (see Table 2).

Table 2

Empirical values of the scaled velocity variation  $\sigma$  and the scaled generalized inverse temperature  $\beta$  for 16 density intervals obtained from single-vehicle data of the Dutch two-lane freeway A9, neglecting clearances in front of long vehicles with  $l_i > 7$  m

$\rho$ (veh./km)	$\sigma$ ( $10^{-1}$ )	$\beta$ ( $10^{-5}$ )
[0; 2.5)	1.095	$\approx 0$
[2.5; 5)	0.967	$\approx 0$
[5; 7.5)	0.989	3.998
[7.5; 10)	1.076	7.331
[10; 12.5)	1.038	11.86
[12.5; 15)	0.960	16.36
[15; 17.5)	0.946	152.9
[17.5; 20]	0.902	442.4
[40; 45)	3.967	634.8
[45; 50)	3.731	711.8
[50; 55)	3.329	915.9
[55; 60)	3.300	1046
[60; 65)	2.996	1186
[65; 70)	2.895	1323
[70; 75)	2.647	1358
[75; 80]	2.735	1279

distributions is obtained for the integer parameter  $\alpha = 1$  in congested traffic, which is, for example, compatible with the intelligent driver model (IDM) and perception-based models [15]. Throughout the free traffic regime, we find a good agreement with  $\alpha = 4$ , corresponding to weak interactions (see Figs. 2 and 3). This is not only a strong support of studies questioning a uniform behavior of drivers in all traffic regimes [3,8]. It also offers an interpretation of the mysterious fractional distance-scaling exponent

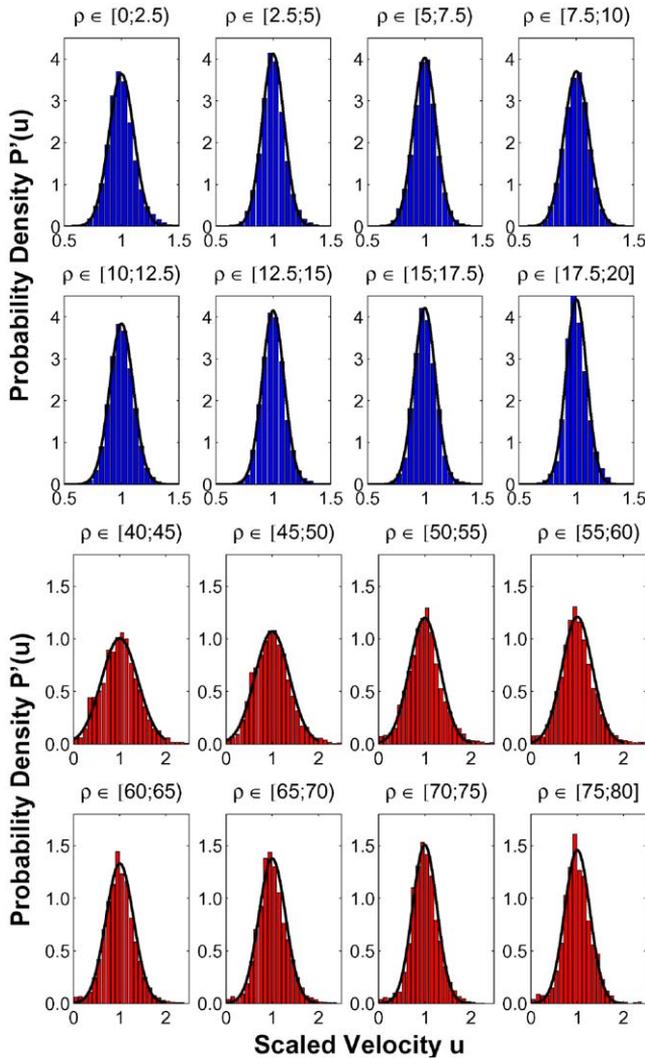


Fig. 1. Scaled velocity distributions for eight density regimes in free traffic (above) and eight density intervals in congested traffic (below). The bar diagrams correspond to the scaled empirical velocity distributions, while the solid curves correspond to the theoretically predicted Gaussian distributions. Note that the mean speeds are always scaled to one, while the variances are given by the scaled empirical values  $\sigma^2$  (see Table 2).

$\alpha + 1 \approx 2.8$  in classical follow-the-leader models [14], which interpolate between the driver behavior in the free and congested regimes.

In summary, we have found that it is successful to generalize thermal-equilibrium properties of a short-ranged power-law Dyson’s gas exposed to a heat reservoir with the scaled generalized inverse temperature  $\beta$  to steady-state vehicle traffic, where  $\beta$  is

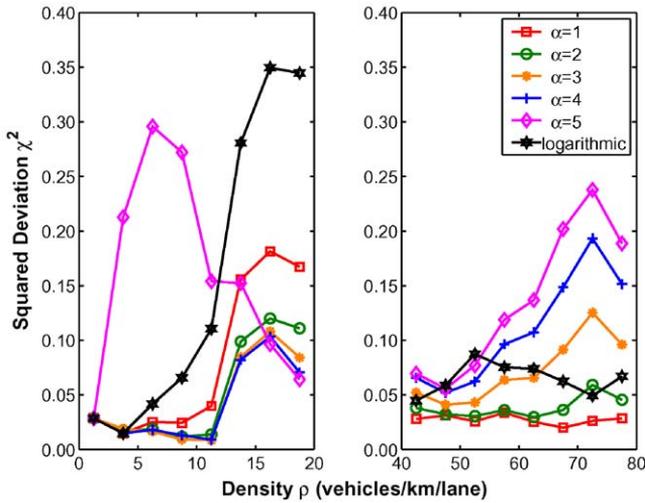


Fig. 2. Sum of squared deviations between the empirical and the theoretical netto distance distributions for various fit parameters  $\alpha$ . The best fit is  $\alpha=4$  throughout the free traffic regime (left) and  $\alpha=1$  throughout the congested regime (right).

an increasing function of the traffic density  $\rho$ . In the regime of congested traffic, this dependency is simply linear:  $\beta(\rho) = 0.0261\rho - 0.4785$ . The presented results show that the shape of the interaction potential of vehicles can be approximated by formula (2) with  $\alpha=4$  for free traffic and 1 for congested traffic. The theoretical predictions are further supported by the Gaussian distribution of the vehicle velocities in all investigated density regimes.

As it is a hard task to derive analytical relations for the clearance distribution (including its normalization constants  $A$  and  $B$ ), we could demonstrate the determination of vehicle interaction potentials only for simple functional relations. Future advances with this method will hopefully allow to determine velocity-dependent interaction forces or functions with turning points, which are desirable to reproduce the dynamical behavior in the unstable traffic regime realistically as well [17]. Determining interaction potentials in freeway traffic does not only contribute to the challenging problem of how to fit time headway or distance distributions of vehicles [18,2,8]. It also advances the quantitative understanding of human behavior. Normally, it is difficult to identify and measure the relevant behavioral variables, and realistic models contain a large number of parameters. Here, progress has been made by powerful methods from statistical physics. Nevertheless, the resulting interaction model is not just a physical model. The new picture of two different regimes with  $\alpha=1$  and 4 points to adaptive driver's behavior to congested and free traffic conditions. The specification of the interaction potential is essential for realistic traffic simulations, which are required for the design of efficient and reliable traffic optimization measures such as intelligent on-ramp controls, driver assistance systems, lane-changing assistants, etc.

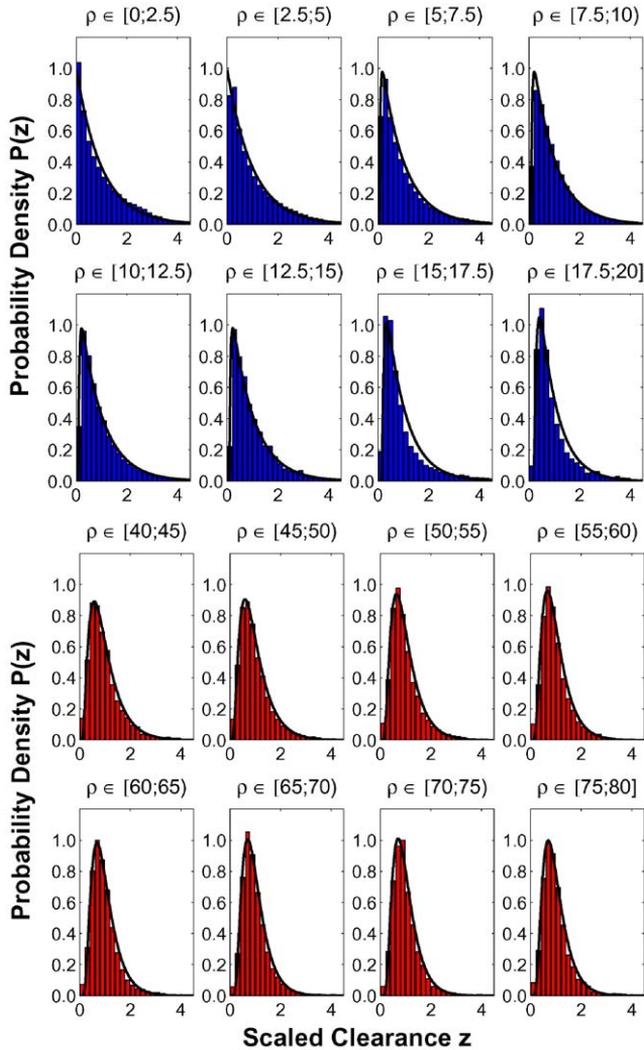


Fig. 3. Distributions of scaled netto distances (clearances)  $z = \rho s$  for eight density intervals in free traffic (above) and eight density regimes in congested traffic (below). The bar diagrams represent the scaled empirical distributions, while the solid curves correspond to the normalized theoretical distributions with the empirically determined values  $\beta$  listed in Table 2 and parameter  $\alpha = 4$  for free traffic, but 1 for congested traffic.

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