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Distance- and Time-headway Distribution for Totally Asymmetric Simple Exclusion Process

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Abstract

We consider the one-dimensional totally asymmetric simple exclusion process on a lattice of N sites with both, the periodic boundary condition and the open boundary condition. Because of its simplicity, this model is often used for highway traffic simulations. Our goal is to investigate the headway distribution for this model in order to compare the microscopic structure of the model with the real highway traffic.

In recent articles, the distance-headway distribution $\wp(d) = \rho(1-\rho)^{d-1}$ for this model has been presented. Using this result, the time-headway distribution $\tau(t)$ is derived, which represents the probability density for the time interval of the length t between the jumps of two successive particles from the observed site. Firstly, the distribution function for the random-sequential time-discrete update is derived, from which the distribution for the time-continuous dynamics can be obtained. Furthermore, the time-headway distribution for forward- and backward-sequential update is studied by means of Monte Carlo simulations. We find out that for all variants of the considered model, the space discrete structure is reflected in the distance-headway distribution. The time-headway distribution reflects (contrary to the real traffic behavior) the "particle-hole symmetry" of the model.

Keywords: totally asymmetric simple exclusion process; vehicular traffic; time-headway distribution

1. Introduction

The attempt to describe and understand the essence of traffic flow dynamics is as old as the traffic itself. In recent decades, the quick development of intelligent technologies enables to simulate a variety of problems by means of computers. Moreover, a new kind of computation by means of the cellular automata has appeared, and it seems to be a very appropriate tool for simulating numerous physical systems including the traffic flow; one of the first cellular models of traffic appeared in Nagel, Schreckenberg (1992). The motivation for simulating the traffic by means of the cellular automata is given by the simplicity of these models, which enables very quick, maybe even real-time,

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simulations of the traffic system. This means we are looking for a model as simple as possible, but complex enough to reflect the "important" aspects of the real traffic system.

One of the simplest models used in the traffic theory is the model based on the totally asymmetric simple exclusion process (TASEP) investigated in this article. This model is a paradigmatic example of non-equilibrium systems with nearest-neighbor interaction. For very elaborate summary of these models see Blythe, Evans (2007). Of course, the TASEP model is too simple to describe fully the complexity of traffic flow; nevertheless, it appears to be an initial point for a variety of more complex but yet analytically solvable systems. Several studies about the next-nearest-neighbor interaction models show a very good agreement with the traffic flow, for more details see (Antal, Schütz, 2000) or (Furtlehner, Lasgoutes, 2009). In Reichenbachl et al. (2007) we may see a two-lane generalization of the TASEP model explaining the traffic jam induction on highways and in Karimipour (1998) a multi-species model with overlapping possibility is investigated.

Our goal in this article is to study the microstructure of the particle interactions represented by the distance- and time-headway distribution bringing insight into the microscopic characteristic of the model. We present the analytical results and compare the microstructure of TASEP with an extensive study of the real traffic microstructure presented in Helbing, Krbálek (2004), Krbálek (2007), and Krbálek, Šeba (2009).

The article is structured as follows: the following section focuses on the traffic theory background of the problem; in the third section, the model is defined; the fourth section summarizes the investigated characteristics of the model; the fifth section deals with the analytical derivation of the time-headway distribution; and the conclusion focuses on the comparison of our results with the traffic flow theory.

2. Investigated traffic flow characteristics

As mentioned above, our investigations are focused predominantly on those properties of cellular models having a realistic interpretation in vehicular traffic streams. As well known from the previous researche (see Derrida et al., 1993), the macroscopic phenomena of TASEP ensembles correspond to the effects observed in freeway samples. Specifically, both systems show similar trends in the so-called fundamental diagram analyzing the dependence of traffic flow on traffic density. Indeed, if comparing the fundamental diagrams in Figure 1 and Figure 3, one can recognize the typical saturation effects (states of high aggregation of particles) in both of them. It means that both systems merge from free phase into the congested phase in which the movement of particles/vehicles is intensively restricted by other elements. This effect is accompanied (in both cases) by the reduction of traffic flow with increasing density.

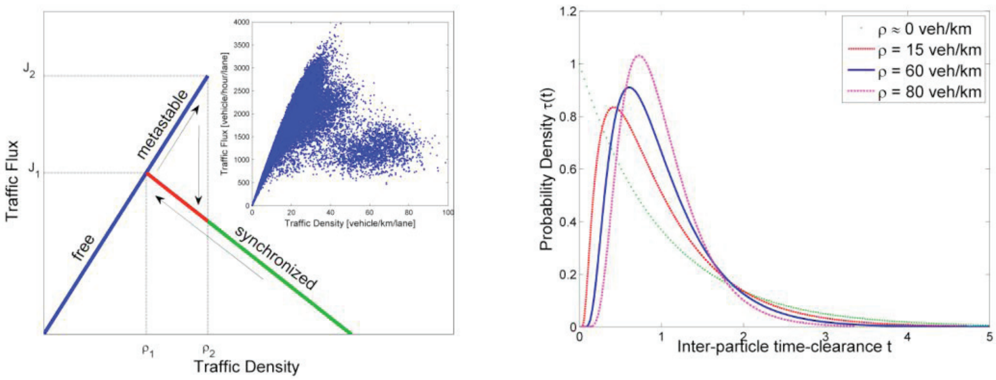

Figure 1: Macroscopic and microscopic traffic flow characteristics. Left graph represents the fundamental diagram of the real traffic sample together with the schematic representation as the mirror image of the letter λ . Right graph represents the scaled time-headway distribution for 4 different densities of cars.

As understandable, the detected transitions between traffic phases can be disclosed in the microscopic structure as well. The detailed statistical analyses of traffic data obtained on European freeways (and associated analytical calculations) have distinctly revealed that probability density for netto-time intervals (refered to as time-clearance or time-headway) between succeeding cars can be satisfactorily approximated by the one-parametric family of functions

$$\tau(t) = A\theta(t)\exp\left(-\tfrac{\beta}{t} - Bt\right), \tag{1}$$

where the symbol $\theta(t)$ corresponds to the Heaviside step-function, and two constants

$$B = \beta + \tfrac{3-\exp\left(-\sqrt{\beta}\right)}{2}, \qquad A^{-1} = 2\tfrac{\beta}{B} K_1(2\sqrt{\beta B}) \tag{2}$$

assure the proper normalization $\sum \tau(t) = 1$ and the scaling to the mean interval $\langle t \rangle = 1$. The function $K_1(x)$ represents the modified Bessel function of second kind and of first order. Whereas for the traffic states of small densities the corresponding time-headway distribution is almost exponential, i.e., $\beta \approx 0$, for saturated states the parameter β is increasing with traffic density. In these cases the non-zero value of β causes the descent in probability for occurrence of two particles close to each other. It means that for the corresponding probability density it holds that $\lim_{t\to 0+} \tau(t) = 0$. The progress in time-headway distribution is visualized in the right graph of Figure 1.

Here we note that the corresponding rescaled-distance-headwaydistribution is, in the zero approximation, of the same shape as the time-headway distribution (see Krbálek, 2010). Also we add that all details concerning the realistic traffic measurements and their evaluations discussed in this article are circumstantially depicted in the section Data analysis in the article (Krbálek, Helbing, 2004).

3. Definition of the TASEP model

Consider a one-dimensional lattice containing N equivalent cells $\{(1),(2),\ldots,(N)\}$. Each cell may be either occupied by a single particle or empty. Particles are moving along the lattice in one direction, usually from left to right, jumping to the neighboring cell. Those jumps are driven by the exclusion process with general parameter $p \in \langle 0,1 \rangle$ in the following way: a particle occupying the cell (i) waits a certain time depending on the dynamics (time-continuous or time-discrete), and then it jumps to the cell $(i+1)$ with probability p if the target cell is empty.

Commonly, two different boundary conditions are distinguished: the open boundary condition and the periodic boundary condition. The open boundary means that a new particle can enter the lattice jumping into the first cell (1) with probability $\alpha \in \langle 0,1 \rangle$ if the cell is empty and a particle can leave the lattice jumping out of the last cell (N) with probability $\beta \in \langle 0,1 \rangle$. The periodic boundary means that the lattice is closed, creating a circle. This means that the right neighbor of the last cell (N) is the first cell (1), i.e., we use the formalism $(i + j \equiv \mathrm{mod}\, N)$, with notation (0)=(N). Both of the boundary conditions are schematically demonstrated in Figure 2.

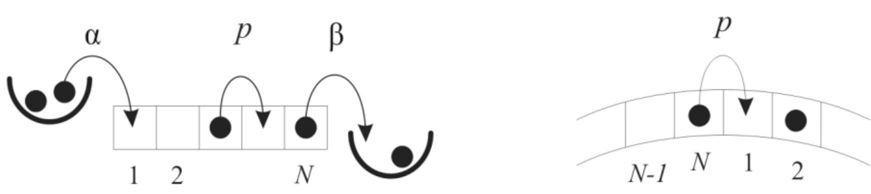

Figure 2: Schematic demonstration of the dynamics of TASEP with open boundaries (left) and periodic boundaries (right).

When considering the time-continuous dynamics, the jumps of each particle are driven by the Poisson process with parameter 1, i.e., the number $\xi_s - \xi_t$ of how many times the particle is "asked" to jump during the time interval $\langle t, s \rangle$ has the Poisson distribution

$$\Pr\left(\xi_s - \xi_t = j\right) = \frac{(s-t)^j}{j!} e^{s-t} . \tag{3}$$

This means that the time the particle waits before it tries to jump, is exponentially distributed with parameter 1. Let us define a stochastic process $\{\eta(t) = (\eta_1(t), \eta_2(t), \dots, \eta_N(t)) \mid t \in T\}$, where $T = \langle 0, +\infty)$ and

$$\eta_i(t) = \begin{cases} 1 & \text{if the cell (i) is occupied,} \\ 0 & \text{if the cell (i) is empty.} \end{cases} \tag{4}$$

As we can see in, e.g., (Blythe, Evans 2007), the stationary distribution $P_N(\eta)$ for probability of finding the system in the state η in the open boundary case can be calculated as

$$P_N(\eta_1, \eta_2, \dots, \eta_N) = \langle W | \prod_{i=1}^{N} [\eta_i E + (1-\eta_i) D] | V \rangle \left(\langle W | (D+E)^N | V \rangle \right)^{-1}, \tag{5}$$

where D, E are square matrices and $\langle W |$, $| V \rangle$ vectors fulfilling $pDE = D + E$, $\alpha \langle W | E = \langle W |$ and $D | V \rangle \beta = | V \rangle$, and in the periodic boundary case

$$P_N(\eta_1, \eta_2, \dots, \eta_N) = \frac{(N-M)!M!}{N!}, \tag{6}$$

where $M = \sum \eta_i$. Note that M is the total number of particles in the lattice, which remains constant; the relation (6) implies that in the periodic case all configurations are equally probable. Due to the simple stationary distribution, we will work further in this article with the lattice with periodic boundaries, although when considering the open boundary case, very similar results have been obtained when investigating the system far enough from the boundaries (for more details see Derrida et al. (1993), Rajewski et al. (1998), Krbálek, Hrabák, (2010)).

For computer simulations, it is more appropriate to use the time-discrete modification of the process. Several time-discrete update procedures have been studied elaborately in (Rajewski et al., 1998), where we can find a solution of the stationary distribution for four different time-discrete approaches together with the density-flow relations for both, open boundary and periodic boundary condition; for a brief summary see also (Blythe, Evans, 2007). Holding the terminology in those articles, we will be interested in *random-sequential, forward-* and *backward-sequential,* and *fully-parallel* updating procedure.

The random-sequential update is, in fact, a discrete realization of the time-continuous dynamics. In every step one of the cells $(i) \in \{(1), (2), \dots, (N)\}$ is chosen at random, and then the exclusion rule is applied to the transition $(i) \to (i+1)$, i.e., if $\eta_i(t) = 1 \wedge \eta_{i+1}(t) = 0$, the particle jumps from (i) to $(i+1)$ with probability p ($\eta_i(t+h) = 0$, $\eta_{i+1}(t+h) = 1$). One step of the update corresponds to $h = 1/N$ of the time unit. This rescaling is necessary when comparing the time-dependent variables measured on two samples with a different lattice size N. In this article, we will respect the following notation: when talking about the number of discrete steps, we will use the letter k ($k, k_0, k_1, \dots \in \{0,1,2,\dots\}$), and in the case of the time measured in time units the letter t ($t, t_0, \dots \in T$) will be used. In the case of the random-sequential update, the set T of allowed times is lattice-size-dependent and it reads $T_N = \{t = \frac{k}{N} \mid k = 0,1,2,\dots\}$. As we may see in (Rajewski et al. 1998), the stationary solution of this process is of exactly the same form as the one for the time continuous process given by (5) or (6). Furthermore, for the count distribution of how many times the particle has been asked to jump it holds that

$$\Pr\left(\xi_s - \xi_t = j\right) = \binom{k_N}{j} \frac{1}{N^j} \left(1 - \frac{1}{N}\right)^{k_N - j} \xrightarrow{N \to +\infty} \frac{(s-t)^j}{j!} e^{s-t}, \text{ where } \frac{k_N}{N} \le s - t < \frac{k_N + 1}{N} . \tag{7}$$

Therefore, it is mathematically correct to use the random-sequential dynamics for investigation and simulation of the time-continuous process. Furthermore, it is easy to show that in those two cases the parameter p only rescales the time and has no other influence on the dynamics; hence, without loss of generality we may set $p=1$.

The last three updates mentioned above represent the parallel updating approach. This means that every particle in the lattice is asked to jump during one step of the procedure and therefore $T=\{t=k \mid k=0,1,..\}$. In the fully parallel update procedure the exclusion rule is applied to all cells simultaneously during one step. It is the most frequently studied of parallel updates. This dynamics has been described in (Nagel, Schreckenberg, 1992) as the Nagel-Schreckenberg model with maximum speed $v_{\max}=1$ and breaking parameter $q=1-p$. The principle of the *sequential* updates is following: the exclusion rule is applied to the cells sequentially in the order $(1)\to(2),(2)\to(3),\ldots,(N-1)\to(N)$ in the forward-sequential case and in the order $(N-1)\to(N),(N-2)\to(N-1),\ldots,(1)\to(2)$ in the backward-sequential case. For the parallel updates the value of jumping rate p is crucial for the dynamics of the process and significantly influences the model behavior.

4. Traffic flow characteristics of the model

Let us now summarize the flow characteristics of the TASEP. The space-discrete nature of the model allows us to define the quantities in the following way: the density ρ_i is understood as the average occupation the of the cell (i), i.e.,

$$\rho_i=\langle\eta_i\rangle=\sum_{\eta}\eta_i P_N(\eta_1,\eta_2,...,\eta_N)\,. \tag{8}$$

The particle flow Q_i through te cell (i) will be calculated as

$$Q_i=\frac{\langle\eta_i(1-\eta_{i+1})\rangle}{\langle\Delta t\rangle}=p\sum_{\eta}\eta_i(1-\eta_{i+1})P_N(\eta_1,\eta_2,...,\eta_N)\,, \tag{9}$$

where $\langle\Delta t\rangle=1/p$ is the mean value of the time a particle in the cell (i) waits before it jumps to the empty cell $(i+1)$. Furthermore, let $\wp_i(d)$ be the probability that the distance between the particle in the cell (i) and the closest particle (in forward direction) is d cells. The distance d is measured from center of one particle to centre of the second particle, i.e., there are $d-1$ empty cells between the cell (i) and the closest occupied one. The value $\wp_i(d)$ is calculated from

$$\wp_i(d)=A\sum_{\eta}\eta_i(1-\eta_{i+1})(1-\eta_{i+2})...(1-\eta_{i+d-1})\eta_{i+d}P_N(\eta_1,\eta_2,...,\eta_N), \tag{10}$$

where A is the normalization constant assuring that $\sum\wp_i(d)=1$. Note that in the large N limit $A=\rho_i^{-1}$.

Consider now the time-continuous (or random sequential) dynamics on the circle of the length N containing M particles. Let $p=1$. From the steady state probability distribution (6) we easily obtain

$$\rho_i=\binom{N-1}{M-1}\binom{N}{M}^{-1}=\frac{M}{N}=\rho\,, \qquad Q_i=\binom{N-2}{M-1}\binom{N}{M}^{-1}=\frac{M}{N}\left(1-\frac{M-1}{N-1}\right)\to\rho(1-\rho)\,, \tag{11}$$

where the limit is meant in the sense $M,N\to\infty$ fulfilling $\rho N\le M<\rho N+1$, e.g., $M(N)=\lfloor\rho N\rfloor$. The flow-density relation is then

$$Q(\rho)=\rho(1-\rho)\,. \tag{12}$$

Investigating the headway distribution $\wp_i(d)$, for d fixed we obtain

$$\wp_i(d) = A\binom{N-d-1}{M-2}\binom{N}{M}^{-1} = \frac{M}{N}\frac{M-1}{N-1}\left(1-\frac{M-2}{N-2}\right)\left(1-\frac{M-2}{N-3}\right)\cdots\left(1-\frac{M-2}{N-d}\right) \to \rho(1-\rho)^{d-1}. \qquad (13)$$

When investigating the system far from the boundaries, i.e., $1 << i << N$, $i+d << N$, the relations (12) and (13) hold true even for the open boundary case, where the letter ρ represents the so called bulk density of the lattice given by

$$\rho(\alpha,\beta) = \begin{cases} \alpha & \alpha<\beta, \alpha<1/2\ , \\ 1-\beta & \beta<\alpha, \beta<1/2\ , \\ 1/2 & \alpha\geq 1/2, \beta\geq 1/2\ . \end{cases} \qquad (14)$$

For the derivation by means of the matrix-product-ansatz (5) of the flow-density relation see (Derrida et al. 1993) or (Blythe, Evans 2007), for the derivation of the distance headway distribution see (Krbálek, Hrabák 2010).

The parallel time-discrete updating procedures have been elaborately examined in (Rajewski et al. 1998) and (Chowdhury et al. 1998), for a brief summary see also (Blythe, Evans 2007). For illustration purposes, we present the flow-density and headway distribution-density relations. For the backward-sequential update it has been shown that

$$Q(p,\rho) = p\rho\frac{1-\rho}{1-p\rho}, \qquad \wp(d) = \rho(1-\rho)^{d-1}, \qquad (15)$$

for the forward-sequential it holds that

$$Q(p,\rho) = p\frac{\rho(1-\rho)}{1-p(1-\rho)}, \qquad \wp(d) = \rho(1-\rho)^{d-1}, \qquad (16)$$

And for the fully-parallel update we have

$$Q(p,\rho) = py \qquad \wp(d) = \frac{y^2}{\rho(1-\rho)}\left(1-\frac{y}{1-\rho}\right)^{d-2}, d\geq 2 \ \text{ and } \ \wp(d) = 1-\frac{y}{\rho}, d=1\ , \qquad (17)$$

where $y=\left(1-\sqrt{1-4p\rho(1-\rho)}\right)/2p$. For illustration, the fundamental diagrams for all the updates are plotted in the left graph of Figure 3.

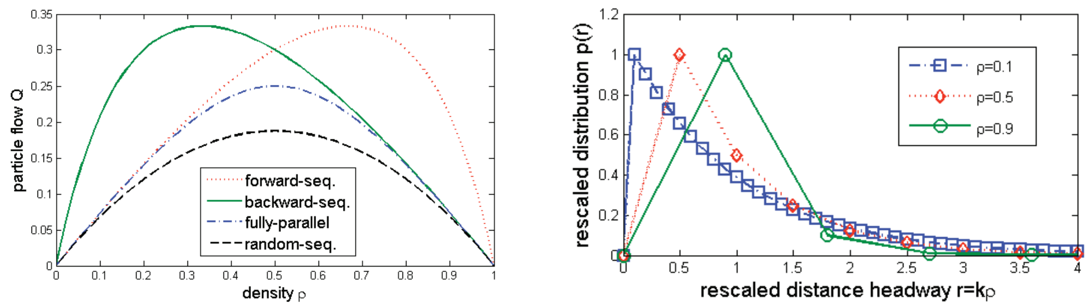

Figure 3: Macroscopic and microscopic characteristics of TASEP. Left graph represents Fundamental diagrams of TASEP for random-sequential (black), forward-sequential (green), backward-sequential (red), and fully-parallel (blue) update with p=0.75. Right graph represents the normalized distance-headway distribution of TASEP.

5. Derivation of the time-headway distribution

The goal of this section is to derive the exact formula for probability density function $\tau(t)$ of the time-headway distribution. We will proceed as follows. First, we derive the discrete probability function $f_N(k)$ of the step-headway distribution for the random-sequential dynamics. Then, the respective time-headway distribution function $F_N(t)$ will be obtained from the relation

$$F_N(t)=\sum_{k=2}^{k(N)} f_N(k)\,,\qquad \text{where } k(N) \text{ fulfils} \qquad \tfrac{k(N)}{N}\le t<\tfrac{k(N)+1}{N}\,. \tag{18}$$

This means that we rescale the time in the sense mentioned in the previous section (k steps of the update correspond to $t=k/N$ time units), so we can compare the distribution functions $F_N(t)$ in a reasonable way. The sequence $k(N)$ can be chosen, e.g., as $k(N)=\lfloor tN\rfloor$. This enables us to calculate the point limit $F(t)=\lim_{N\to+\infty}F_N(t)$. As shown below, the limit distribution function $F(t)$ is absolutely continuous, and, therefore, the respective density function can be calculated via $f(t)=\frac{dF(t)}{dt}$. As the process with random-sequential update converges, in the sense mentioned in Section 3, to the process with time-continuous dynamics, we can conclude that the desired time headway distribution has the limit density $f(t)$. To compare the results with the normalized distribution (1), we will rescale the time, so

$$\tau(t_r)=\langle t\rangle f(t)\,,\ t_r=t/\langle t\rangle\,. \tag{19}$$

Let us now calculate the values $f_N(k)$ for $k=1,2,\ldots$. We use the same structure as in (Chowdhury et al. 1998). We are interested in the step-headway k between the leading particle (LP) and the following particle (FP), $k=k_{LP}-k_{FP}$, where $k_P, P\in\{LP,FP\}$ is the time (in steps) in which the particle P passes (jumps out of) the reference cell (RC). The probability that the FP leaves the reference cell (RC) exactly k steps after the LP jumped from (RC) to (RC+1) will be calculated as

$$f_N(k)=\sum_{k_1=1}^{k-1} q_N(k-k_1\,|\,k_1)\,p_N(k_1)\,, \tag{20}$$

where $p_N(k_1)$ denotes the probability that the FP enters the cell (RC) exactly k_1 steps after the LP jumped out of (RC) and $q_N(k-k_1\,|\,k_1)$ stands for the conditional probability that the FP waits exactly $k-k_1$ steps in the cell (RC) before jumping out of it if the FP entered the cell (RC) exactly k_1 steps after the LP jumped out of it. As we can see, it holds that

$$p_N(k_1)=\sum_{d=1}^{k_1}\wp(d)w(d,k_1)=\sum_{d=1}^{k_1}\binom{k_1-1}{d-1}\rho(1-\rho)^{d-1}\,\frac{1}{N^d}\left(1-\frac{1}{N}\right)^{k_1-d}=\frac{\rho}{N}\left(1-\frac{\rho}{N}\right)^{k_1-1}, \tag{21}$$

where

$$w(a,b)=\binom{b-1}{a-1}\frac{1}{N^a}\left(1-\frac{1}{N}\right)^{b-a} \tag{22}$$

denotes the probability of the particle (FP in this case) being chosen $(a-1)$-times during $b-1$ steps and being chosen in the b-th step. Because the neighboring cell of the FP is always empty, it means that the FP jumps to the neighboring cell any time it is be chosen.

The conditional probability $q_N(k-k_1\,|\,k_1)$ can be calculated as

$$q_N(k_2\,|\,k_1)=v_N(k_1)w(1,k_2)+\big(1-v_N(k_1)\big)\cdot\sum_{k=1}^{k_2-1}u_N(k)w(1,k_2-k)\,, \tag{23}$$

where $u_N(k)$ stands for the probability of the LP leaving the cell (RC+1) exactly k steps after it had jumped from (RC) to (RC+1) and $v_N(k_1)=\sum_{k=1}^{k_1-1}u_N(k)$ denotes the probability of the LP leaving the cell (RC+1) at last k_1-1 steps after it had jumped out of (RC), i.e., $v_N(k_1)$ is the probability that during the k_1-th step the cell (RC+1) will be empty ($\eta_{RC+1}(k_{LP}/N+k_1/N)=0$). Now, when observing the movement of holes through particles, the probability

$u_N(k)$ corresponds to the probability that it will last k steps after the jump of the LP from (RC) to (RC+1) before a hole approaches the cell (RC+1), traveling backwards. Hence, from the particle-hole symmetry it follows that

$$u_N(k) = \tfrac{\sigma}{N}\left(1-\tfrac{\sigma}{N}\right)^{k-1}, \tag{24}$$

where for convenience we use the notation $\sigma = 1-\rho$. Therefore,

$$v_N(k_1) = \sum_{k=1}^{k_1-1} u_N(k) = 1-\left(1-\tfrac{\sigma}{N}\right)^{k_1-1}. \tag{25}$$

Substituting (24) and (25) into (23) we obtain

$$\begin{aligned} q_N(k_2 \mid k_1) &= \left(1-\left(1-\tfrac{\sigma}{N}\right)^{k_1-1}\right)\tfrac{1}{N}\left(1-\tfrac{1}{N}\right)^{k_2-1} + \left(1-\tfrac{\sigma}{N}\right)^{k_1-1}\sum_{k=1}^{k_2-1}\tfrac{\sigma}{N}\left(1-\tfrac{\sigma}{N}\right)^{k_1-1}\left(1-\tfrac{1}{N}\right)^{k_2-k-1}\tfrac{1}{N} = \\ &= \underbrace{\tfrac{1}{N}\left(1-\tfrac{1}{N}\right)^{k_2-1}}_{q_{1,N}(k_2|k_1)} - \underbrace{\tfrac{1}{\rho N}\left(1-\tfrac{\sigma}{N}\right)^{k_1-1}\left(1-\tfrac{1}{N}\right)^{k_2-1}}_{q_{2,N}(k_2|k_1)} - \underbrace{\tfrac{\sigma}{\rho N}\left(1-\tfrac{\sigma}{N}\right)^{k_2+k_1-2}}_{q_{3,N}(t_2|t_1)}. \end{aligned} \tag{26}$$

The equation (20) then reads

$$\begin{aligned} f_N(k) &= \sum_{k_1=1}^{k-1} q_{1,N}(k-k_1 \mid k_1)\,p_N(k_1) + \sum_{k_1=1}^{k-1} q_{2,N}(k-k_1 \mid k_1)\,p_N(k_1) + \sum_{k_1=1}^{k-1} q_{3,N}(k-k_1 \mid k_1)\,p_N(k_1) = \\ &= \tfrac{\rho}{\sigma N}\left[\left(1-\tfrac{\rho}{N}\right)^{k-1} - \left(1-\tfrac{1}{N}\right)^{k-1}\right] + \tfrac{1}{\rho\sigma}\left[\left(1-\tfrac{1}{N}\right)^{k-1} - \left(1-\tfrac{\rho}{N}\right)^{k-1}\left(1-\tfrac{\sigma}{N}\right)^{k-1}\right] + \tfrac{\sigma}{\rho N}\left(1-\tfrac{\sigma}{N}\right)^{k-2}\left[1-\left(1-\tfrac{\rho}{N}\right)^{k-1}\right]. \end{aligned} \tag{27}$$

Now we can calculate the values of the time-headway distribution function $F_N(t)$ according to equation (18). We note that the distribution function will be calculated as the sum $F_N(t) = F_{N,1}(t) + F_{N,2}(t) + F_{N,3}(t)$, where

$$\begin{aligned} F_{1,N}(t) &= \tfrac{\rho}{\sigma N}\sum_{k=2}^{\lfloor tN \rfloor}\left(1-\tfrac{\rho}{N}\right)^{k-1} - \tfrac{\rho}{\sigma N}\sum_{k=2}^{\lfloor tN \rfloor}\left(1-\tfrac{1}{N}\right)^{k-1} = \tfrac{1}{\sigma}\left[1-\left(1-\tfrac{\rho}{N}\right)^{\lfloor tN \rfloor}\right] - \tfrac{\rho}{\sigma}\left[1-\left(1-\tfrac{1}{N}\right)^{\lfloor tN \rfloor}\right] \\ &\xrightarrow{N\to+\infty} \tfrac{1}{\sigma}\left(1-e^{-\rho t}\right) - \tfrac{\rho}{\sigma}\left(1-e^{-t}\right), \end{aligned} \tag{28}$$

$$\begin{aligned} F_{2,N}(t) &= \tfrac{1}{\rho\sigma}\sum_{k=2}^{\lfloor tN \rfloor}\left(1-\tfrac{1}{N}\right)^{k-1} - \tfrac{1}{\rho\sigma}\sum_{k=2}^{\lfloor tN \rfloor}\left(1-\tfrac{\rho}{N}\right)^{k-1}\left(1-\tfrac{\sigma}{N}\right)^{k-1} = \\ &= \tfrac{N}{\rho\sigma}\left\{\left(1-\tfrac{1}{N}\right)^{\lfloor tN \rfloor} - \left(\tfrac{N}{N-\rho\sigma}\right)\left[1-\left(1-\tfrac{\rho}{N}\right)^{\lfloor tN \rfloor}\left(1-\tfrac{\sigma}{N}\right)^{\lfloor tN \rfloor}\right]\right\} \xrightarrow{N\to+\infty} e^{-t}(1+t)-1, \end{aligned} \tag{29}$$

$$\begin{aligned} F_{3,N}(t) &= \tfrac{\sigma}{\rho N}\sum_{k=2}^{\lfloor tN \rfloor}\left(1-\tfrac{\sigma}{N}\right)^{k-2} - \tfrac{\sigma}{\rho N}\sum_{k=2}^{\lfloor tN \rfloor}\left(1-\tfrac{\sigma}{N}\right)^{k-2}\left(1-\tfrac{\rho}{N}\right)^{k-1} = \tfrac{1}{\rho}\left[1-\left(1-\tfrac{\sigma}{N}\right)^{\lfloor tN \rfloor-1}\right] - \\ &- \tfrac{\sigma}{\rho}\,\tfrac{N-\rho}{N-\sigma\rho}\left[1-\left(1-\tfrac{\sigma}{N}\right)^{\lfloor tN \rfloor-1}\left(1-\tfrac{\rho}{N}\right)^{\lfloor tN \rfloor-1}\right] \xrightarrow{N\to+\infty} \tfrac{1}{\rho}\left(1-e^{-\sigma t}\right) - \tfrac{\sigma}{\rho}\left(1-e^{-t}\right). \end{aligned} \tag{30}$$

Taken together, for $t \geq 0$ we obtain

$$F(t) = \lim_{N\to+\infty} F_N(t) = \tfrac{1}{\sigma}\left(1-e^{-\rho t}\right) - \tfrac{\rho}{\sigma}\left(1-e^{-t}\right) + \tfrac{1}{\rho}\left(1-e^{-\sigma t}\right) - \tfrac{\sigma}{\rho}\left(1-e^{-t}\right) + e^{-t}(1+t) - 1, \tag{31}$$

which leads to

$$f(t) = F'(t) = \tfrac{\rho}{\sigma}\left(e^{\sigma t}-1\right)e^{-t} + \tfrac{\sigma}{\rho}\left(e^{\rho t}-1\right)e^{-t} - te^{-t}. \tag{32}$$

It is easy to verify that $\langle t \rangle = \int_0^{+\infty} t f(t)dt = \frac{1}{\rho\sigma}$. After rescaling the time ax according to (19) we obtain

$$\tau(t_r) = \frac{1}{\sigma^2} e^{-t_r/\sigma} + \frac{1}{\rho^2} e^{-t_r/\rho} - \left(\frac{1}{\sigma^2} + \frac{1}{\rho^2} + \frac{t_r}{\rho^2\rho^2}\right) e^{-t_r/\rho\sigma}. \qquad (33)$$

Several examples of the distribution densities $f(t)$ and $\tau(t_r)$ are plotted in Figure 4.

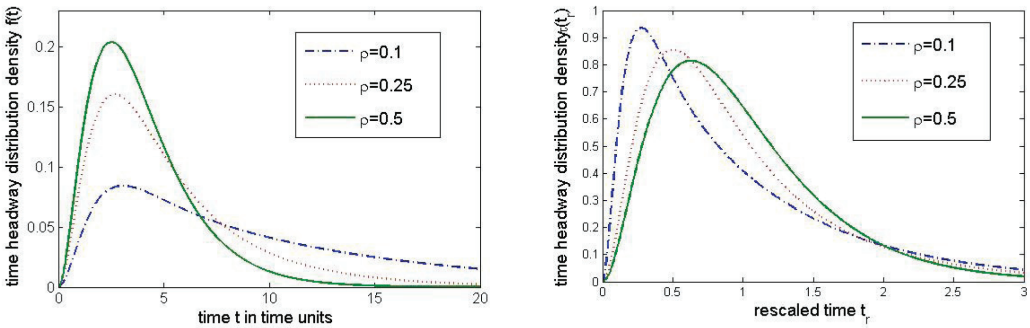

Figure 4: Time-headway distribution $f(t)$ and rescaled time-headway distribution $\tau(t_r)$ for the process with time continuous dynamics.

Let us now investigate the time-headway distribution for time-discrete parallel updating procedures. In this case, thee step-headway distribution is identical to the time-headway distribution. The fully-parallel case has been analytically studied in (Chowdhury et al. 1998) as the Nagel-Schreckenberg model with maximum speed $v_{\max} = 1$ and the breaking parameter $q = 1 - p$. Using the site-oriented mean field theory the authors have derived the distribution $f_\rho(t),\ t = 1, 2, \ldots$ in the form

$$f_\rho(t) = \frac{py}{\rho - y}\left(1 - \frac{py}{\rho}\right)^{t-1} + \frac{py}{\sigma - y}\left(1 - \frac{py}{\sigma}\right)^{t-1} - \left(\frac{py}{\rho - y} + \frac{py}{\sigma - y}\right)(1-p)^{t-1} - p^2(t-1)(1-p)^{t-2}, \qquad (34)$$

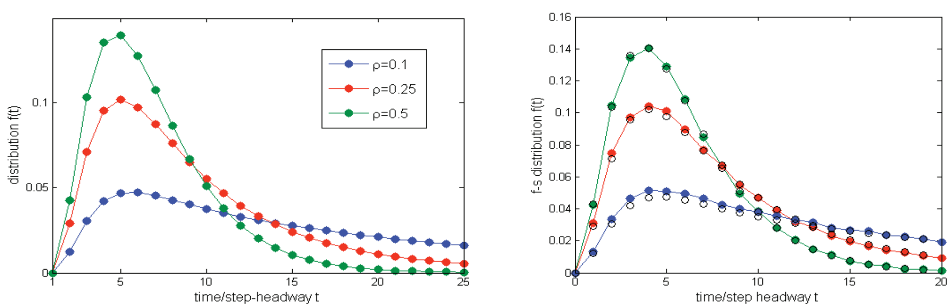

Figure 5: Time headway distribution $f_\rho(t)$ for fully-parallel update (left), and forward-sequential update (right) with $p = 0.5$. The distribution for the forward-sequential update is plotted for densities ρ_c (green), $\rho_c / 2$ (red), and $\rho_c / 5$ (blue), where ρ_c denotes the density of maximum flow.

where $y = \left(1 - \sqrt{1 - 4p\rho(1-\rho)}\right)/2p$. As we can see, the distribution shows the expected symmetry in density ρ, $f_\rho(t) = f_\sigma(t)$; this symmetry is caused, analogically as in (32), by the particle-hole symmetry of the dynamics. This symmetry is broken by the sequential updating procedures. Nevertheless, it is easy to observe another symmetry in the motion of the particles. The forward-sequential "particle" dynamics with the density ρ corresponds to the backward-sequential "holes" dynamics with the density $\sigma = 1 - \rho$. This symmetry is reflected in the time-headway distribution as well. The numerical study of the quantity is presented in Figure 5. We can see that the density ρ_c

corresponding to the most narrow distribution $f_{\rho_c}(t)$ corresponds to the maximum flow density, i.e., $\rho_c = \left(1-\sqrt{1-p}\right)/p$ in the backward-sequential case, and $\rho_c = 1-\left(1-\sqrt{1-p}\right)/p$ in the forward-sequential case.

6. Conclusion

We have studied the microscopic characteristics of the totally asymmetric simple exclusion process - the headway distribution. Considering the distance-headway distribution, the power-law dependence $\wp(d) = \rho(1-\rho)^{d-1}$ has been derived in (Krbálek, Hrabák 2010), reflecting the space-discrete nature of the model. As a consequence, the comparison of the normalized distribution (see Figure 3) with the real traffic distribution (1) (see Figure 1) is not conclusive. On the other hand, the time-headway distribution density $\tau(t_r)$ for the time-continuous dynamics (see equation (33) and Figure 4) gives a deep insight in the microstructure of the dynamics. Comparing the distribution densities (1) and (33), we conclude that for the density $\rho \approx 0$ the distribution (1) is close to the exponential distribution; for the densities fulfilling $0 < \rho < 1/2$, the distribution is Poisson-like and the shape shows the same trend as the shape of the distribution (1). On the other hand, due to the particle-hole symmetry of the model, the distribution for the density $\sigma = 1-\rho$ is the same as the one for density ρ. That means, the distribution for densities greater than $1/2$ does not correspond to the trend observed in the real traffic studies (see Section 2).

Inspired by the macroscopic results for the backward- or forward-sequential updating procedures, for which the particle-hole symmetry is broken leading to the asymmetry of the fundamental diagram, we studied the time-headway distribution of these modifications by means of the computer simulations. Although the particle-hole symmetry does not hold, the distribution shows analogical behavior as the one for the time-continuous dynamics. It has the same trend-changing at the value $\rho = \rho_c$. That means, contrary to the fundamental diagram, the time-discrete updating procedure does not lead to a better agreement of the model behavior with the real traffic flow. We may observe, that the capacity of a highway stream (80 veh./km) corresponds to the density $\rho \approx 1/2$ of the occupation of the lattice. Therefore, it would be beneficial to modify the model in the way that will maintain the shape of the distribution and its dependence on the density, but significantly changes the flow-density dependence. One of the proposed solutions is an addition of a stronger next-nearest-neighbor interaction (Antal, Schütz 2000), maintaining the analytical solvability of the problem.

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