Homework 01

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MacDonald functions (modified second order *Bessel* functions) $K_a(x)$ for $a \in \mathbb{R}$ are defined as:

$$x^a K_a(x) = 2^{a-1} \int_0^\infty y^{a-1} \exp\left(-\frac{x^2}{4y}\right) \exp(-y) \, \mathrm{d}y \tag{*}$$

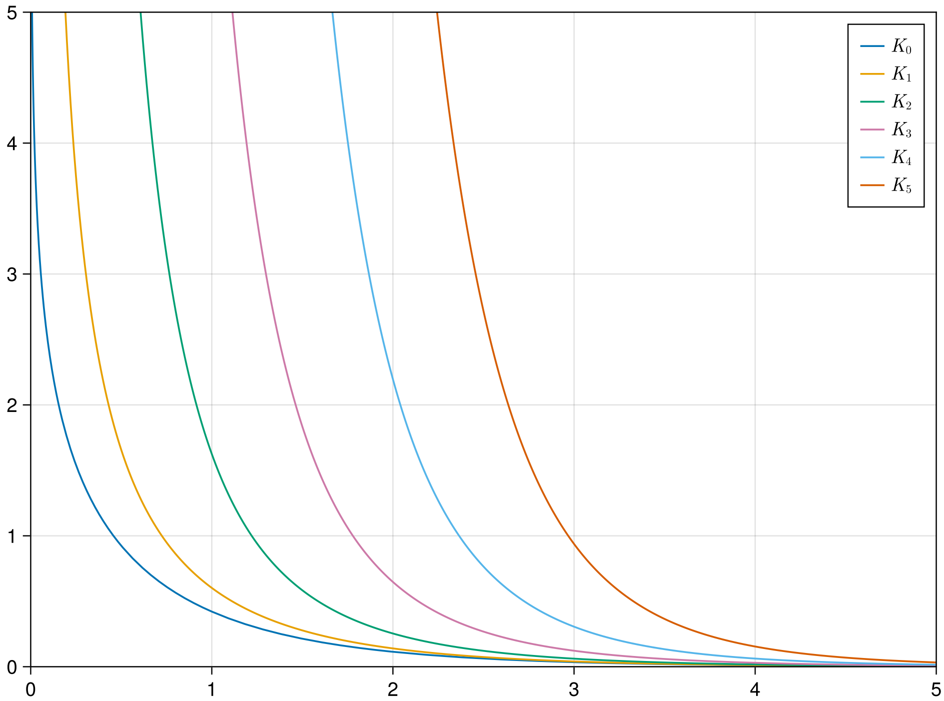

Figure 1: *MacDonald* functions for $a \in \{0, 1, 2, 3, 4, 5\}$

1. Prove the following reccurent relationships between *MacDonald* functions:

$$K_{a-1}(x) - K_{a+1}(x) = -\frac{2a}{x} K_a(x) \tag{ii}$$

$$K_a{}'(x) = -K_{a-1}(x) - \frac{a}{x} K_a(x) \tag{iii}$$

2. Prove the following reccurent relationships between *MacDonald* functions:

$$K_{a-1}(x) + K_{a+1}(x) = -2K_a{}'(x) \tag{iv}$$

$$K_a{}'(x) = -K_{a+1}(x) + \frac{a}{x} K_a(x) \tag{v}$$

3. Derive all moments of the normal distribution. That is derive

$$\mu_{2k}(N(\mu, \sigma)) = \frac{(2k)!}{(2k)!!} \cdot \sigma^{2k} \tag{vi}$$

and the odd moments.

Problems 1 and 2

It is advantageous to approach both of the first two tasks at the same time, because the system of 4 equations presented by them is overspecified. Let us therefore first select the set of independent equations that will prove the whole system. If we add equations iii and v, we get

$$\begin{aligned} 2{K_a}'(x) &= -K_{a-1}(x) - \frac{a}{x}K_a(x) - K_{a+1}(x) + \frac{a}{x}K_a(x) \\ 2{K_a}'(x) &= -K_{a-1}(x) - K_{a+1}(x) \\ -2{K_a}'(x) &= K_{a-1}(x) + K_{a+1}(x), \end{aligned}$$

which is equation iv. Therefore we get equation v ∧ equation iii ⇒ equation iv. Further, we can add equations ii and iii to get

$$\begin{aligned} -K_{a+1}(x) + \frac{a}{x}K_a(x) &= -\frac{2a}{x}K_a(x) + {K_a}'(x) \\ -K_{a+1}(x) - \frac{a}{x}K_a(x) &= {K_a}'(x), \end{aligned}$$

which is equation iii. Therefore we get equation ii ∧ equation iii ⇒ equation v. In conclusion, we only need to prove equations ii and iii to get the whole set.

Proof of Equation ii

Our objective is to prove

$$K_{a-1}(x) - K_{a+1}(x) = -\frac{2a}{x}K_a(x).$$

We start by rewriting the integral

$$a\int_0^\infty y^{a-1}\exp\left(-\frac{x^2}{4a}\right)\exp(-y)\,\mathrm{d}y$$

using integration by parts

$$\int_0^\infty f'(y)g(y)\,\mathrm{d}y = [f(y)g(y)]_0^\infty - \int_0^\infty f(y)g'(y)\,\mathrm{d}y$$

and by setting

$$\begin{aligned} f(y) &= y^a \\ g(y) &= \exp\left(-\frac{x^2}{4y}\right)\cdot\exp(-y) \end{aligned}$$

for which the derivatives and the constant is

$$\begin{aligned} f'(y) &= ay^{a-1} \\ g'(y) &= \frac{x^2}{4y^2}\exp\left(-\frac{x^2}{4y}\right)\exp(-y) - \exp\left(-\frac{x^2}{4y}\right)\exp(-y) \\ [f(y)g(y)]_0^\infty &= 0, \end{aligned}$$

where we can use for instance L'Hopitals rule to justify the constant. We can recognise the left-hand side $\int_0^\infty f'(y)g(y)\,\mathrm{d}y$ of the by parts rule to be our original integral. Therefore we have

$$a\int_0^\infty y^{a-1}\exp\left(-\frac{x^2}{4a}\right)\exp(-y)\,\mathrm{d}y =$$
$$= \int_0^\infty y^{a}\exp\left(-\frac{x^2}{4y}\right)\exp(-y) - \frac{x^2}{4}\int_0^\infty y^{a-2}\exp\left(-\frac{x^2}{4y}\right)\exp(-y)\,\mathrm{d}y$$

Multiplying both sides by $-2^a\,/\,x^{a+1}$ and leaving the term $-\frac{2a}{x}$ from the final form of equation ii aside, gives us

$$-\frac{2a}{x}\frac{2^{a-1}}{x^a}\int_0^\infty y^{a-1}\exp\left(-\frac{x^2}{4a}\right)\exp(-y)\,\mathrm{d}y =$$
$$= \frac{2^{a-2}}{x^{a-1}}\int_0^\infty y^{a-2}\exp\left(-\frac{x^2}{4y}\right)\exp(-y)\,\mathrm{d}y - \frac{2^a}{x^{a+1}}\int_0^\infty y^{a}\exp\left(-\frac{x^2}{4y}\right)\exp(-y)\,\mathrm{d}y$$

and we can recognise the terms of the final form of equation ii using definition *

$$-\frac{2a}{x}K_a(x) = K_{a-1}(x) - K_{a+1}(x)$$

and this concludes our proof. ∎

Proof of Equation iii

Our objective is to prove

$$K_a{}'(x) = -\,K_{a-1}(x) - \frac{a}{x}K_a(x).$$

Starting with the left-hand side of the definition relation * for a *MacDonald* function of the order a and making a derivative with respect to the variable x, leads us to

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^a K_a(x)] = ax^{a-1}K_a(x) + x^a K_a{}'(x).$$

We continue by making the same derivative with respect to x of the right hand side, but first we need to check, whether it is possible to swap the order of integrating and differentiating, that is if the following holds

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[2^{a-1}\int_0^\infty y^{a-1}\exp\left(-\frac{x^2}{4y}\right)\exp(-y)\,\mathrm{d}y\right] = 2^{a-1}\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}x}\left[y^{a-1}\exp\left(-\frac{x^2}{4y}\right)\exp(-y)\right]\mathrm{d}y =$$
$$= 2^{a-1}\int_0^\infty y^{a-1}\exp\left(-\frac{x^2}{4y}\right)\exp(-y)\left(-\frac{x}{2y}\right)\mathrm{d}y = -\,2^{a-2}x\cdot\int_0^\infty y^{a-2}\exp\left(-\frac{x^2}{4y}\right)\exp(-y)\,\mathrm{d}y.$$

The conditions for this swap are the existence of the partial derivative of the integrand almost everywhere and it being continuous which is evident and the existence of an integrable function of y dominating the absolute value of the derivative. But since $-\frac{x^2}{4y} \le 0$, we have $0 < \exp\left(-\frac{x^2}{4y}\right) < 1$ and our integrand is absolutely dominated by $g(y) = y^{a-2}\exp(-y) \le y^{a-2}$ and therefore is integrable. We end up with the equation

$$ax^{a-1}K_a(x) + x^a K_a{}'(x) = -\,2^{a-2}x \cdot \int_0^\infty y^{a-2} \exp\left(-\frac{x^2}{4y}\right) \exp(-y)\,\mathrm{d}y$$

and after dividing both sides by x^a we can recognise the *MacDonald* function of order $a-1$ on the right hand side giving us

$$\frac{a}{x}K_a(x) + K_a{}'(x) = -\,K_{a-1}(x)$$

which is equation iii. ∎

Problem 3

We start by writing the probability distribution function of the normal distribution

$$p(x \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\left(\frac{x-\mu}{\sqrt{2\sigma^2}}\right)^2\right).$$

The raw and central moments of a random variable $X \sim N(\mu, \sigma)$ are defined to be

$$\begin{aligned} \mu'_n &= E[X^n] \\ \mu_n &= E[(X - EX)^n]. \end{aligned}$$

Using the linearity of the expected value (E) and the binomial expansion, we can rewrite the central moment as

$$\mu_n = \sum_{k=0}^{n} \binom{n}{k} (-EX)^k E\left[X^{n-k}\right] = \sum_{k=0}^{n} \binom{n}{k} (-\mu'_1)^k \mu'_{n-k},$$

therefore it is sufficient to only calculate the raw moments. If we write the definition expression for the raw moments and manipulate it a little we can also rewrite it in terms of simpler objects like so

$$\begin{aligned} \mu'_n &= \int_{\mathbb{R}} x^n p(x \mid \mu, \sigma)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} x^n \exp\left(-\left(\frac{x-\mu}{\sqrt{2\sigma^2}}\right)^2\right) \mathrm{d}x = \\ &= \left\{ y := \left(\frac{x-\mu}{\sqrt{2\sigma^2}}\right);\ \sqrt{2\sigma^2}\,\mathrm{d}y = \mathrm{d}x;\ x = \left(\sqrt{2\sigma^2} \cdot y + \mu\right) \right\} = \\ &= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \left(\sqrt{2\sigma^2} \cdot y + \mu\right)^n \exp(-y^2)\,\mathrm{d}y = \\ &= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \left\{ \sum_{k=0}^{n} \binom{n}{k} \left(\sqrt{2\sigma^2} \cdot y\right)^k \cdot \mu^{n-k} \right\} \cdot \exp(-y^2)\,\mathrm{d}y = \\ &= \frac{1}{\sqrt{\pi}} \cdot \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} \cdot \left(\sqrt{2\sigma^2}\right)^k \int_{\mathbb{R}} y^k \exp(-y^2)\,\mathrm{d}y. \end{aligned}$$

At this point we can clearly recognise the Gaussian integral $\int_{\mathbb{R}} x^k \exp(-x^2)\,\mathrm{d}x$ that appears in the sum. We can calculate these integrals recursively. As a side note, we can see that for k odd, the integrand is an odd function and therefore the whole integral is equal to 0. Using the notation

$$I_k := \int_{\mathbb{R}} x^k \exp(-x^2)\,\mathrm{d}x$$

we use the expansion through integration by parts as

$$
\begin{aligned}
I_k &= \int_{\mathbb{R}} x^k \exp(-x^2)\,\mathrm{d}x = \\
&= \left\{ f'(x) = x^k;\ g(x) = \exp(-x^2);\ f(x) = \frac{x^{k+1}}{k+1};\ g'(x) = -\,2x\exp(-x^2) \right\} = \\
&= \left[\frac{x^{k+1}}{k+1} \cdot \exp(-x^2) \right]_{-\infty}^{+\infty} + \frac{2}{k+1} \int_{\mathbb{R}} x^{k+2} \exp(-x^2)\,\mathrm{d}x = \\
&= 0 + \frac{2}{k+1} \cdot I_{k+2} = \frac{2}{k+1} \cdot I_{k+2}.
\end{aligned}
$$

We got to the recurrent relation

$$
I_{k+2} = \frac{k+1}{2} \cdot I_k
$$

for Gaussian integrals.

The only thing that is left to resolve for the solution to the whole system of integrals $\{I_{2n} \mid n \in \mathbb{N}\}$ is I_0 (keep in mind that the odd integrals I_k are 0). We can do that in many ways. One way would be by entering the complex plane, parametrizing a Cauchy rectangle curve that includes the x axis in the limit and using Cauchy's theorem and the symmetry between the integrals along the rectangle sides to show that $I_0 = \sqrt{\pi}$. A second way to do this would be by making the transition to polar coordinates (this proof is originally by the famous Simeon Denis Poisson)

$$
\begin{aligned}
I_0 &= \int_{\mathbb{R}} \exp(-x^2)\,\mathrm{d}x = 2 \cdot \int_0^{\infty} \exp(-x^2)\,\mathrm{d}x = 2 \cdot \sqrt{\int_0^{\infty} \int_0^{\infty} \exp(-(x^2+y^2))\,\mathrm{d}x\,\mathrm{d}y} = \\
&= 2 \cdot \sqrt{\int_0^{\frac{\pi}{2}} \int_0^{\infty} \exp(-r^2) \cdot r\,\mathrm{d}r\,\mathrm{d}\theta}.
\end{aligned}
$$

It should be easy to fill in the blanks left after the demonstrated transition.

Another way is to use the normalisation of the normal distribution as a fact and realising that

$$
\begin{aligned}
I_0 &= \int_{\mathbb{R}} \exp(-x^2)\,\mathrm{d}x = \left\{ y := x \cdot \left(\sqrt{2\sigma^2}\right);\ \sigma \equiv \sqrt{\frac{1}{2}} \right\} = \\
&= \sqrt{2\sigma^2} \int_{\mathbb{R}} \exp\left(-\left(\frac{y}{\sqrt{2\sigma^2}} \right)^2 \right) \mathrm{d}y = \sqrt{2\pi\sigma^2} \cdot \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\left(\frac{y}{\sqrt{2\sigma^2}} \right)^2 \right) \mathrm{d}y = \\
&= \sqrt{2\pi\sigma^2} \cdot \left\| p\left(x \mid \mu = 0, \sigma = \sqrt{\frac{1}{2}} \right) \right\|_{L_1} = \sqrt{2\pi\sigma^2} = \sqrt{\pi}.
\end{aligned}
$$

However, if we look at the formulation of the problem, we see that even this is not necessary, since the relation we want to prove includes σ^2 which is the second central moment and it should could be easily written in terms of I_2 and continuing from there. Although this would be an elegant way to dodge the need to provide a formula for the Gaussian integrals, we will use the raw moments and therefore this would be unecessarily difficult. Capitalising on the recursive identity with the knowledge of the value of I_0 we get

$$I_{2n} = I_0 \cdot \prod_{k=0}^{n} \frac{2k-1}{2} = \frac{(2n-1)!!}{2^n} \cdot \sqrt{\pi}.$$

We left the computation of moments at the point

$$\mu'_n = \frac{1}{\sqrt{\pi}} \cdot \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} \cdot \left(\sqrt{2\sigma^2}\right)^k \cdot I_k$$

and using the fact that for the odd indexes $I_k = 0$ we get

$$\mu'_n = \frac{1}{\sqrt{\pi}} \cdot \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n}{2k} \mu^{n-2k} \cdot \left(\sqrt{2\sigma^2}\right)^{2k} \cdot I_{2k} = \frac{1}{\sqrt{\pi}} \cdot \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n}{2k} \mu^{n-2k} \cdot \left(2\sigma^2\right)^k \cdot I_{2k} =$$
$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n}{2k} \mu^{n-2k} \cdot \sigma^{2k} (2k-1)!!.$$

It is beneficial to make the further calculations for odd and even moments separately. Also we will simplify by calculating moments for a centred normal distributed random variables (i.e. $\mu = 0$). Therefore we have $X \sim N(0, \sigma)$. The relation for odd moments translates for this specific situation to

$$\mu'_{2n+1} = \sum_{k=0}^{n} \binom{2n+1}{2k} 0^{(2n+1)-2k} \cdot \sigma^{2k} (2k-1)!! = 0$$

and for the even moments we have

$$\mu'_{2n} = \sum_{k=0}^{n} \binom{2n}{2k} 0^{2n-2k} \cdot \sigma^{2k} (2k-1)!! = \binom{2n}{2n} \sigma^{2n} (2n-1)!! = \sigma^{2n} (2n-1)!! = \frac{(2n)!}{(2n)!!} \sigma^{2n},$$

which is the relation that we wanted to prove. There is one more cheap generalisation, where we can realise, that if $X \sim N(\mu, \sigma)$ then $Y := (X - \mu) = (X - EX) \sim N(0, \sigma)$ and therefore, if we calculate the central moment of X, we get

$$\mu_n(X) = E\left[(X - EX)^n\right] = E[Y^n] = \mu'_n(Y) = \begin{cases} 0 \text{ if } (n \bmod 2) \equiv 1 \\ \frac{(2n)!}{(2n)!!} \sigma^{2n} \text{ if } (n \bmod 2) \equiv 0 \end{cases}.$$