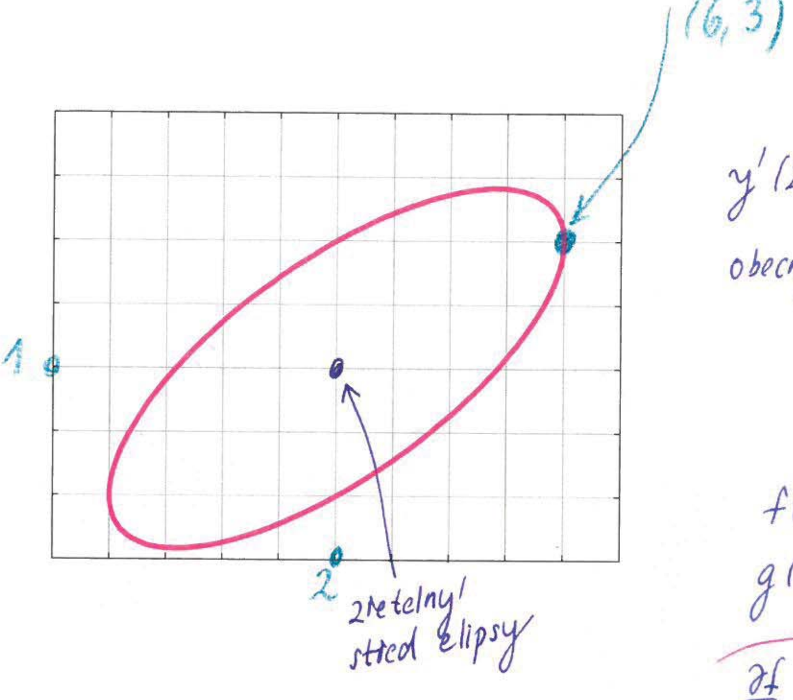
8b



$y'(2y-x)+x-y-1=0$ & $y(6)=3$

obecný tvar ODR:

$$f(x,y)+g(x,y)\cdot y'=0$$
$$y(x_0)=y_0$$

$$f(x,y)=x-y-1$$
$$g(x,y)=2y-x$$

$\frac{\partial f}{\partial y}=-1$ & $\frac{\partial g}{\partial x}=1 \Rightarrow$ jde o exaktní ODR

$$H(x,y)=\int_{x_0}^{x} f(s,y)\,ds+\int_{y_0}^{y} g(x_0,t)\,dt=\int_{6}^{x}(s-y-1)\,ds+\int_{3}^{y}(2t-6)\,dt=$$

$$=\left[\frac{s^2}{2}-ys-s\right]_6^x+\left[t^2-6t\right]_3^y=\frac{x^2}{2}-xy-x-18+6y+6+$$

$$+y^2-6y-9+18=\frac{x^2}{2}-xy-x+y^2-3=0$$

Formální řešení: $\underline{x^2-2xy+2y^2-2x=6}$

$A=\begin{pmatrix}1&-1\\-1&2\end{pmatrix}$ $\det A=1$ $\vec{b}=\begin{pmatrix}1\\0\end{pmatrix}$

výpočet středu: $A\cdot\vec{s}=\vec{b}$ $\begin{pmatrix}1&-1\\-1&2\end{pmatrix}\begin{pmatrix}s_1\\s_2\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$

$$s_1=\begin{vmatrix}1&-1\\0&2\end{vmatrix}=2 \quad \& \quad s_2=\begin{vmatrix}1&1\\-1&0\end{vmatrix}=1$$

$$\underline{\vec{s}=(2,1)}$$

↪ když vím, že střed má souřadnice (2,1), je už snadné vykreslit bod (6,3)

← za správnou lokaci bodu (6,3)

$$\sum_{n=1}^{\infty} (-1)^n \frac{(4n)!!}{(4n+1)!!} \frac{\cosh(nx)+\sinh(nx)}{\cosh(nx)} \qquad SK \text{ na } \langle 0,+\infty)$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{(4n)!!}{(4n+1)!!} \frac{e^{nx}+e^{-nx}+e^{nx}-e^{-nx}}{e^{nx}+e^{-nx}} = 2\sum_{n=1}^{\infty}(-1)^n \frac{(4n)!!}{(4n+1)!!} \underbrace{\frac{1}{1+e^{-2nx}}}_{\text{úprava}}$$

W-kritérium nelze užít, neboť

$$\left|2(-1)^n \frac{(4n)!!}{(4n+1)!!}\frac{1}{1+e^{-2nx}}\right| \leq 2\frac{(4n)!!}{(4n+1)!!}\sup_{x\in\mathbb{R}^+}\left|\frac{1}{1+e^{-2nx}}\right| = \underbrace{2\frac{(4n)!!}{(4n+1)!!}}_{\text{majoranta}}$$

ale $\sum \frac{(4n)!!}{(4n+1)!!}$ je divergentní

$$\Leftarrow \lim_{n\to\infty} n\left(1-\frac{(4n+4)!!}{(4n+5)!!}\frac{(4n+1)!!}{(4n)!!}\right) = \lim_{n\to\infty} n\left(1-\frac{(4n+4)(4n+2)}{(4n+5)(4n+3)}\right) =$$

$$= \lim_{n\to\infty} n\frac{32n+15-24n-8}{(4n+5)(4n+3)} = \frac{8}{16} = \frac{1}{2} \in (0;1)$$

$$\underbrace{\text{– ale řada } \sum_{n=1}^{\infty}(-1)^n\frac{(4n)!!}{(4n+1)!!} \text{ tedy konverguje (alternativní Raabe)}}$$

Abelovo kritérium

$$f_n(x) = (-1)^n\frac{(4n)!!}{(4n+1)!!} \qquad g_n(x) = \frac{2}{1+e^{-2nx}}$$

a) $\sum f_n(x)$ konverguje stejnoměrně ($\Leftarrow$ jde o číselnou posloupnost)

b) $|g_n(x)| = \left|\frac{2}{1+e^{-2nx}}\right| \leq 2$ omezenost

c) monotonie

$$\frac{1}{1+e^{-2nx}} < \frac{1}{1+e^{-(2n+2)x}}$$

$$e^{-2nx}e^{-2x} < e^{-2nx}$$

$$e^{-2x} < 1 \quad \checkmark$$

SK na $\langle 0;+\infty)$

10 ~~bodů~~

10

(+ 1 bod bonus)

Řešte diferenciální rovnici

$$xy''' + (3-6x)y'' + 12(x-1)y' + (12-8x)y = 24x - 8x^2 - 12,$$

~~víte-li, že rovnici řeší jakákoli funkce~~

$$\cancel{y(x) = \mathbb{C}xe^{2x} + x}, \qquad (\mathbb{C} \in \mathbf{R}).$$

partikulární řešení známe $\Rightarrow$ stačí řešit $L(y) = 0$

$w_1, w_2 \in \Omega_q$

$\Rightarrow w_1 - w_2 \in \Omega_0 \quad \& \quad w_1 - w_2 = xe^{2x}$

máme jednu funkci z Ω_0 (hurá :-))

$$y = zxe^{2x}$$
$$y' = (z'x + z + 2zx)e^{2x}$$
$$y'' = (z''x + 2z' + 4z'x + 4z + 4zx)e^{2x}$$
$$y''' = (z'''x + 3z'' + 6z''x + 12z' + 12z'x + 12z + 8zx)e^{2x}$$

pokud někdo užil tu samou substituci $y = z \cdot e^{2x}$, neměl to moc podložené $\Rightarrow$ $\Rightarrow$ body se neuznávají

$$z'''x^2 + 3z''x + 6z''x^2 + 12xz' + 12z'x^2 + \cancel{12xz} + 8zx^2 +$$
$$+ 3z''x + 6z' + 12z'x + \cancel{12z} + 12xz - 6z''x^2 - 12z'x - 24z'x^2 -$$
$$- 24zx - \cancel{24zx^2} + 12z'x^2 + \cancel{12zx} + \cancel{24zx^2} - 12z'x - \cancel{12z} - \cancel{24zx} + 12zx - 8zx^2 = 0$$

$$x^2z''' + 6xz'' + 6z' = 0 \qquad w = \frac{dz}{dx}$$

$$x^2w'' + 6xw' + 6w = 0$$

$$x = e^t \;\dots\; t = \ln(x) \qquad \frac{dw}{dx} = \frac{dw}{dt}\frac{1}{x} \qquad \frac{d^2w}{dx^2} = -\frac{1}{x^2}\dot{w} + \frac{1}{x^2}\ddot{w}$$

$$\ddot{w} + 5\dot{w} + 6w = 0$$

$$\lambda^2 + 5\lambda + 6 = (\lambda+3)(\lambda+2) = 0 \qquad F_S = \{e^{-3t}; e^{-2t}\}$$

$$w(t) = C_1e^{-3t} + C_2e^{-2t}$$

toto platí i pro $x<0$!

$$w(x) = \frac{C_1}{x^3} + \frac{C_2}{x^2}$$

$$z(x) = \int w(x)\,dx = \frac{D_1}{x^2} + \frac{D_2}{x} + D_3$$

$$y(x) = \frac{D_1}{x}e^{2x} + D_2e^{2x} + D_3xe^{2x} + x \;; \quad I = \mathbb{R}^+$$

$x = -e^t$

$$\underbrace{y(x) = \frac{D_1}{x}e^{2x} + D_2e^{2x} + D_3xe^{2x} \;; \quad I = \mathbb{R}_-}$$

bonus 1 bod pro toho, kdo řešil i verzi $x<0$

$g(x) = \ln(1+ax) \qquad a>0$

$$\frac{d^k g}{dx^k} = \frac{a^k}{(1+ax)^k}(-1)^{k+1}\cdot(k-1)! \qquad \frac{d^k g}{dx^k}(0) = a^k(-1)^{k+1}(k-1)!$$

Řadou podezřelou z toho, že je Maclaurinovou řadou funkce $g(x)$ je:

$$\sum_{k=1}^{\infty} a^k(-1)^{k+1}\frac{(k-1)!}{k!}x^k = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{a^k}{k}x^k$$

$$R^{-1} = \lim_{k\to+\infty}\frac{a^{k+1}}{k+1}\cdot\frac{k}{a^k} = a \quad\Rightarrow\quad R = 1/a$$

Zjevně: $\quad U = \left(-\frac{1}{a}; \frac{1}{a}\right\rangle$

Označme: $\quad s(x) \overset{\Delta}{=} \sum_{k=1}^{\infty}(-1)^{k+1}\frac{a^k}{k}x^k \qquad \mathrm{Dom}(s) = \left(-\frac{1}{a}; \frac{1}{a}\right\rangle$

Čemu se rovná $s(x)$?

$$s'(x) = \sum_{k=1}^{\infty}(-1)^{k+1}a^k x^{k-1} = \frac{a}{1+ax}$$

$$\Rightarrow s(x) = \ln(1+ax) + \mathcal{H} \;\wedge\; \underbrace{\mathcal{H} = 0 \Leftarrow s(0) = 0}$$

← musí to tam být!

Protože jsme zjistili, že $s(x) = g(x)$, je skutečně nalezená řada Maclaurinovou řadou funkce $\ln(1+ax)$.

$\Rightarrow \ln(1+ax)$ je analytická v nule

*) pokud někdo použil známého rozvoje funkce $\ln(1+x)$, nezískává žádný bod ⇐ rozpor se zadáním

Na $\mathcal{H} = C(\langle -1, 1\rangle)$ u standardní skalární součin definuji takto: 6b

$$\langle f|g\rangle := \int_{-1}^{1} f(x)\,g(x)\,dx$$

– na Hilbertově prostoru také platí, že

$$\|f\|^2 = \langle f|f\rangle = \int_{-1}^{1} f^2(x)\,\cancel{g}\,dx$$

$$\alpha_1 = \arccos \frac{|\langle f|g\rangle|}{\|f\|\cdot\|g\|} \quad \& \quad \alpha_2 = \arccos \frac{\langle h|g\rangle}{\|h\|\cdot\|g\|}$$

$$f(x) = x^2 \quad \& \quad g(x) = x^4 \quad \& \quad h(x) = x^6$$

$$\langle f|g\rangle = \int_{-1}^{1} f(x)\,g(x)\,dx = \int_{-1}^{1} x^6\,dx = \frac{2}{7}$$

$$\|f\|^2 = \int_{-1}^{1} f^2(x)\,dx = \int_{-1}^{1} x^4\,dx = \frac{2}{5}$$

$$\langle h|g\rangle = \int_{-1}^{1} h(x)g(x)\,dx = \int_{-1}^{1} x^{10}\,dx = \frac{2}{11}$$

$$\|h\|^2 = \int_{-1}^{1} h^2(x)\,dx = \int_{-1}^{1} x^{12}\,dx = \frac{2}{13}.$$

$$\frac{|\langle f|g\rangle|}{\|f\|\cdot\|g\|} < \frac{|\langle h|g\rangle|}{\|h\|\cdot\|g\|}$$

$$|\langle f|g\rangle|^2\cdot\|\cancel{g}h\|^2 < |\langle h|g\rangle|^2\cdot\|f\|^2$$

$$\frac{4}{49}\cdot\frac{2}{13} < \frac{4}{121}\cdot\frac{2}{5}$$

$$121\cdot 5 < 49\cdot 13$$

$$605 < 637$$

$\Rightarrow$ funkce $x^{\cancel{4}\,2}$ svírá s funkcí $x^{\cancel{6}\,4}$ větší úhel než funkce $x^{\cancel{4}\,6}$

$\Leftarrow$ arccos x je klesající funkce

$$s(x) = \sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!} = 1 + \frac{1}{4!}x^4 + \frac{1}{8!}x^8 + \dots = 1 + \frac{1}{24}x^4 + \frac{1}{8!}x^8 + \dots$$ 9b

$$\Rightarrow \quad s(0)=1 \ \wedge \ s'(0)=0 \ \wedge \ s''(0)=0 \ \wedge \ s'''(0)=0 \ \wedge \ s^{(4)}(0)=1 \ \wedge \dots$$

$$s'(x) = \sum_{n=1}^{\infty} \frac{x^{4n-1}}{(4n-1)!} \qquad s''(x) = \sum_{n=1}^{\infty} \frac{x^{4n-2}}{(4n-2)!} \qquad s'''(x) = \sum_{n=1}^{\infty} \frac{x^{4n-3}}{(4n-3)!}$$

$$s''''(x) = \sum_{n=1}^{\infty} \frac{x^{4n-4}}{(4n-4)!} = \sum_{m=0}^{\infty} \frac{x^{4m}}{(4m)!} \quad \Rightarrow \quad s'''' = s$$

Cauchyova úloha:

$$s'''' - s = 0 \quad \wedge \quad s(0)=1,\ s'(0)=0,\ s''(0)=0,\ s'''(0)=0$$

$$\lambda^4 - 1 = (\lambda^2-1)(\lambda^2+1) = (\lambda-1)(\lambda+1)(\lambda-i)(\lambda+i) = 0$$

$$F_J = \{ e^x, e^{-x}, \cos x, \sin x \}$$

$$s(x) = C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x \qquad (C_1, C_2, C_3, C_4) = ?$$

$$s'(x) = C_1 e^x - C_2 e^{-x} - C_3 \sin x + C_4 \cos x$$

$$s''(x) = C_1 e^x + C_2 e^{-x} - C_3 \cos x - C_4 \sin x$$

$$s'''(x) = C_1 e^x - C_2 e^{-x} + C_3 \sin x - C_4 \cos x$$

$$\left.\begin{aligned} s(0) &= C_1 + C_2 + C_3 = 1 \\ s'(0) &= C_1 - C_2 + C_4 = 0 \\ s''(0) &= C_1 + C_2 - C_3 = 0 \\ s'''(0) &= C_1 - C_2 - C_4 = 0 \end{aligned}\right\} \quad 2C_3 = 1 \quad \Big\} \quad C_4 = 0 \qquad \begin{aligned} C_1 &= C_2 \\ 2C_2 + \tfrac{1}{2} &= 1 \\ C_2 &= \tfrac{1}{4} \\ C_1 &= \tfrac{1}{4} \end{aligned}$$

$$(C_1, C_2, C_3, C_4) = (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2}, 0)$$

$$s(x) = \frac{1}{4}e^x + \frac{1}{4}e^{-x} + \frac{1}{2}\cos x = \frac{1}{2}(\cos x + \cosh x)$$

$$\mathrm{Dom}(s) = G = \mathbb{R} \quad \Big| \Leftarrow \ R^{-1} = \lim_{n\to\infty} \frac{(4n)!}{(4n+4)!} = \lim_{n\to\infty} \frac{1}{(4n+4)(4n+3)(4n+2)(4n+1)} = 0$$

$n^{!!!!} := n(n-4)$ 8

10 bodů

$$f(x) = \frac{1}{\sqrt[4]{7+x}} = (7+x)^{-1/4} \qquad c = 9$$

$$f'(x) = -\frac{1}{4}(7+x)^{-5/4} \qquad f''(x) = \left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)(7+x)^{-9/4}$$

$$\underline{f^{(n)}(x) = (-1)^n \frac{(4n-3)!!!!}{4^n}(7+x)^{-n-\frac{1}{4}}}$$

$$f^{(n)}(9) = (-1)^n \frac{(4n-3)!!!!}{4^n}\left(\frac{1}{16}\right)^{n+\frac{1}{4}} = (-1)^n \frac{(4n-3)!!!!}{2\cdot 4^n}\frac{1}{16^n}$$

$$\underline{f^{(n)}(9) = \frac{(-1)^n}{2}\frac{(4n-3)!!!!}{64^n}}$$

$$\underline{\frac{1}{\sqrt[4]{7+x}} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n!}\frac{(4n-3)!!!!}{64^n}(x-9)^n}$$

$$R^{-1} = \lim_{n\to\infty} \frac{(4n+1)!!!!}{(n+1)!}\frac{64^n}{64^{n+1}}\frac{n!}{(4n-3)!!!!} = \lim_{n\to\infty}\frac{4n+1}{n+1}\frac{1}{64} = \frac{4}{64} = \frac{1}{16}$$

$$\underline{\underline{R = 16}}$$

$$\underline{\underline{I = (-7;\, 25)}}$$

správné určení krajních bodů!

nebo

krajní řady: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2\cdot n!}\frac{(4n-3)!!!!}{64^n}(\mp 1)^n \cdot 16^n = \frac{1}{2}\sum_{n=1}^{\infty}(\pm 1)^n \frac{(4n-3)!!!!}{4^n \cdot n!}$

(znaménko + platí pro levý kraj oboru konvergence)

← tvar krajních řad

$$\lim_{n\to\infty} n\left(1 - \frac{(4n+1)!!!!}{4^{n+1}(n+1)!}\cdot\frac{4^n\cdot n!}{(4n-3)!!!!}\right) = \lim_{n\to\infty} n\left(1-\frac{4n+1}{4(n+1)}\right) =$$

$$= \lim_{n\to\infty} n\left(\frac{4n+4-4n-1}{4n+4}\right) = \lim_{n\to\infty}\frac{3n}{4n+4} = \frac{3}{4} \in (0,1)$$

$$\Rightarrow \quad O = (-7;\, 25\rangle$$

5b

Rozšířená q-forma:

$$q_0(x,y,z) = x^2 - 2\alpha xy + \alpha y^2 + 2\alpha xz - \alpha^2 z^2$$

Lagrangeův algoritmus:

$$q_0(x,y,z) = (x-\alpha y+\alpha z)^2 + \alpha y^2 - \alpha^2 y^2 + 2\alpha^2 yz - 2\alpha^2 z^2 =$$

$$= (x-\alpha y+\alpha z)^2 - 2\alpha^2(z^2 - yz) + (\alpha-\alpha^2)y^2 =$$

$$= (x-\alpha y+\alpha z)^2 - 2\alpha^2\left(z-\frac{y}{2}\right)^2 + \left(\alpha-\alpha^2+2\alpha^2\frac{\cancel{y}}{4}\right)y^2 =$$

$$= \underline{(x-\alpha y+\alpha z)^2 - 2\alpha^2\left(z-\frac{y}{2}\right)^2 + \left(\alpha-\frac{\alpha^2}{2}\right)y^2}$$

Kdy je hlavní signatura, tj. $sg(q_0)$, rovna trojici $(1,1,1)$?

- jistě $\alpha \overset{!}{\neq} 0$, jinak by $sg(q_0) = (1,0,2)$
- ale jeden čtverec z Lagrangeova konzolidovaného tvaru musí vypadnout, protože nulový index setrvačnosti je roven jedné

$$\Rightarrow \underbrace{\alpha - \frac{\alpha^2}{2} \overset{!}{=} 0 \wedge \alpha \overset{!}{\neq} 0}_{\alpha \overset{!}{=} 2}$$

- pro $\alpha = 2$ je naštěstí znaménko druhé členu záporné, takže skutečně: $\alpha = 2 \Rightarrow SG(Q) = (1,1,1)$

$$\underline{\alpha = 2}$$

10b

$$y' = \frac{-x+3y-10}{x+y+2} \quad \Rightarrow \quad \underbrace{x-3y+10}_{f(x,y)} + \underbrace{(x+y+2)}_{g(x,y)}\, y' = 0$$

$$\frac{\partial f}{\partial y} = -3 \quad \& \quad \frac{\partial g}{\partial x} = 1 \qquad -3 \neq 1$$

$$\left.\begin{aligned} -a+3b-10 &= 0 \\ a+b+2 &= 0 \end{aligned}\right\} \Rightarrow \quad a = -\frac{1}{4}\begin{vmatrix} 10 & 3 \\ -2 & 1 \end{vmatrix} = -4 \quad \& \quad b = -\frac{1}{4}\begin{vmatrix} -1 & 10 \\ 1 & -2 \end{vmatrix} = 2$$

$$(a,b) = (-4,2)$$

Substituce:

$$\left.\begin{aligned} x &= z+a = z-4 \\ y &= u+b = u+2 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} -x+3y-10 &= -z+3u \\ x+y+2 &= z+u \end{aligned}$$

$$\& \quad u = u(z)$$

Transformovaná rovnice: $$u' = \frac{-z+3u}{z+u}$$ (a ta je homogenní)

•) explicitní řešení: $u = Cz \quad \Rightarrow \quad C(z+Cz) = (-z+3Cz)$

$$C^2 - 2C + 1 = 0$$

řešení: $u = z \quad \Rightarrow \quad y-2 = x+4 \quad \Rightarrow \quad y = x+6$

••) implicitní řešení: $u(z) = z \cdot w(z) \quad \Rightarrow \quad u' = w + z \cdot w'$

$$w + z \cdot w' = \frac{-1+3\cdot w}{1+w}$$

$$z \cdot w' = \frac{3w-1-w-w^2}{1+w}$$

$$-z \cdot w' = \frac{w^2-2w+1}{1+w} \equiv \frac{(w-1)^2}{1+w}$$

$$\int \frac{1+w}{(w-1)^2}\, dw = -\int \frac{1}{z}\, dz$$

$$\ln|w-1| + \frac{-2}{w-1} = -\ln|z| + C$$

$$|w\cdot z - z| \cdot e^{-2/w-1} = K$$

$$(u-z)\cdot e^{-2z/u-z} = K$$

$$(y-x-6)\cdot e^{-\frac{2x+8}{y-x-6}} = K$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{n!}{(2n+1)!!} \frac{n^2 2^n}{n^2+x^2} \qquad A = \langle 0; +\infty)$$

10 b

$$g_n(x) = \frac{1}{n^2+x^2} \qquad g_n'(x) = -\frac{2x}{n^2+x^2} < 0 \text{ na } A \Rightarrow g_n(x) \text{ je na } A \text{ klesající}$$

a navíc $\lim_{x\to+\infty} g_n(x) = 0 \wedge g_n(x) \geq 0$

$\Rightarrow g_n(x)$ má maximum v bodě $x_0 = 0$

$\Rightarrow |g_n(x)| \leqq 1/n^2$

Pro Weierstrassovo kritérium tedy:

$$\left| (-1)^n \frac{n!}{(2n+1)!!} \frac{n^2\cdot 2^n}{n^2+x^2} \right| \leqq \frac{n!\, 2^n}{(2n+1)!!}$$

Jak ale dopadne konvergence řady $\sum_n \frac{n!\, 2^n}{(2n+1)!!}$?

Raabe:

$$\lim_{n\to\infty} n\left(1 - \frac{(n+1)!\, 2^{n+1}}{(2n+3)!!} \frac{(2n+1)!!}{n!\, 2^n}\right) = \lim_{n\to\infty} n\left(1 - \frac{2n+2}{2n+3}\right) =$$

$$= \lim_{n\to\infty} n \frac{1}{2n+3} = \frac{1}{2} \in (0,1) \dots$$ to je ale pech :(

Znovu a lépe :)

Abelovo kritérium: dělení: $f_n(x) := (-1)^n \frac{n!\, 2^n}{(2n+1)!!}$ & $g_n(x) = \frac{n^2}{n^2+x^2}$

α) posloupnost je skutečně monotónní, neboť

$$g_{n+1}(x) = \underbrace{\frac{(n+1)^2}{(n+1)^2+x^2} \geq \frac{n^2}{n^2+x^2}} = g_n(x)$$

$$(n+1)^2 \cdot n^2 + (n+1)^2 x^2 \geq n^2(n+1)^2 + n^2 x^2$$

$$(n+1)^2 x^2 > n^2 x^2$$

β) stejnoměrná omezenost:

$$|g_n(x)| \leqq n^2 \cdot \frac{1}{n^2} = 1 \quad \text{na } A = \langle 0; +\infty)$$

γ) řada $\sum f_n(x)$ je pouze číselná, tj. stačí dokázat její konvergenci — z výše uvedeného Raabeova kritéria je zřejmé, že řada (díky přítomnosti členu $(-1)^n$) konverguje

α✓, β✓, γ✓ $\Rightarrow$ uvedená řada SK na A

1 (bodů) 10 bodů

Nalezněte maximální řešení diferenciální rovnice

$$y''' - 6y'' + 12y' - 8y = \frac{10}{x}e^{2x}.$$

$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$ ... dvojka je trojnásobný kořen $\Leftarrow (\lambda-2)^3 = \lambda^3 - 6\lambda^2 + 12\lambda - 8$

$\Rightarrow$ fundamentální systém: $\{e^{2x};\ xe^{2x};\ x^2e^{2x}\}$

$$W = \begin{vmatrix} e^{2x} & xe^{2x} & x^2e^{2x} \\ 2e^{2x} & (1+2x)e^{2x} & (2x+2x^2)e^{2x} \\ 4e^{2x} & (4+4x)e^{2x} & (2+8x+4x^2)e^{2x} \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & x & x^2 \\ 2 & 1+2x & 2x+2x^2 \\ 4 & 4+4x & 2+8x+4x^2 \end{vmatrix} =$$

$$= e^{6x}(2+\cancel{8x}+\cancel{4x^2}+\cancel{4x}+16x^2+\cancel{8x^3} + \cancel{8x^2}+8x^3+8x^2+\cancel{8x^3} - \cancel{4x^2} - \cancel{8x^3} - \cancel{8x} - \cancel{8x^2} - 8x^2 - 8x^3 -$$

$$- \cancel{4x} - 16x^2 - \cancel{8x^3}) = 2e^{6x}$$

$$\Delta_1 = \frac{10}{x}e^{2x} \begin{vmatrix} x & x^2 \\ 1+2x & 2x+2x^2 \end{vmatrix} e^{4x} = 10e^{6x}x \begin{vmatrix} 1 & 1 \\ 1+2x & 2+2x \end{vmatrix} = 10xe^{6x}$$

$$\Delta_2 = -\frac{10}{x}e^{2x} \begin{vmatrix} 1 & x^2 \\ 2 & 2x+2x^2 \end{vmatrix} e^{4x} = -10e^{6x} \begin{vmatrix} 1 & x \\ 2 & 2+2x \end{vmatrix} = -20e^{6x}$$

$$\Delta_3 = \frac{10}{x}e^{2x} \begin{vmatrix} 1 & x \\ 2 & 1+2x \end{vmatrix} e^{4x} = \frac{10}{x}e^{6x}$$

$f_1(x) = 10x \cdot \frac{1}{2}$ & $f_2(x) = -20 \cdot \frac{1}{2}$ & $f_3(x) = \frac{10}{x} \cdot \frac{1}{2} = \frac{5}{x}$

$F_1(x) = \frac{1}{2}5x^2 + \alpha$ & $F_2(x) = -10x + \beta$ & $F_3(x) = 5\ln x + \mu$

$$y(x) = \alpha e^{2x} + \beta xe^{2x} + \mu x^2e^{2x} + \frac{5}{2}x^2e^{2x} - 10x^2e^{2x} + 5x^2e^{2x}\ln x$$

$$y(x) = \alpha e^{2x} + \beta xe^{2x} + \mu x^2e^{2x} + 5x^2e^{2x}\ln x$$

$\mathrm{Dom}(y) = (0, +\infty)$

$\rho(\vec{x}, \vec{y}) = 2\lceil |x_1 - y_1| \rceil + |x_2 - y_2|$

$\Rightarrow$ hledáme body $\vec{x} = (x_1, x_2)$, pro které $\rho(\vec{x}, \vec{0}) = 2\lceil |x_1| \rceil + |x_2| < 6$

α) if $x_1 = 0$ then $2\lceil |x_1| \rceil + |x_2| = |x_2| \wedge |x_2| \overset{!}{<} 6$

β) if $|x_1| \in (0, 1\rangle$ then $\lceil |x_1| \rceil = 1$ a my hledáme body, kde

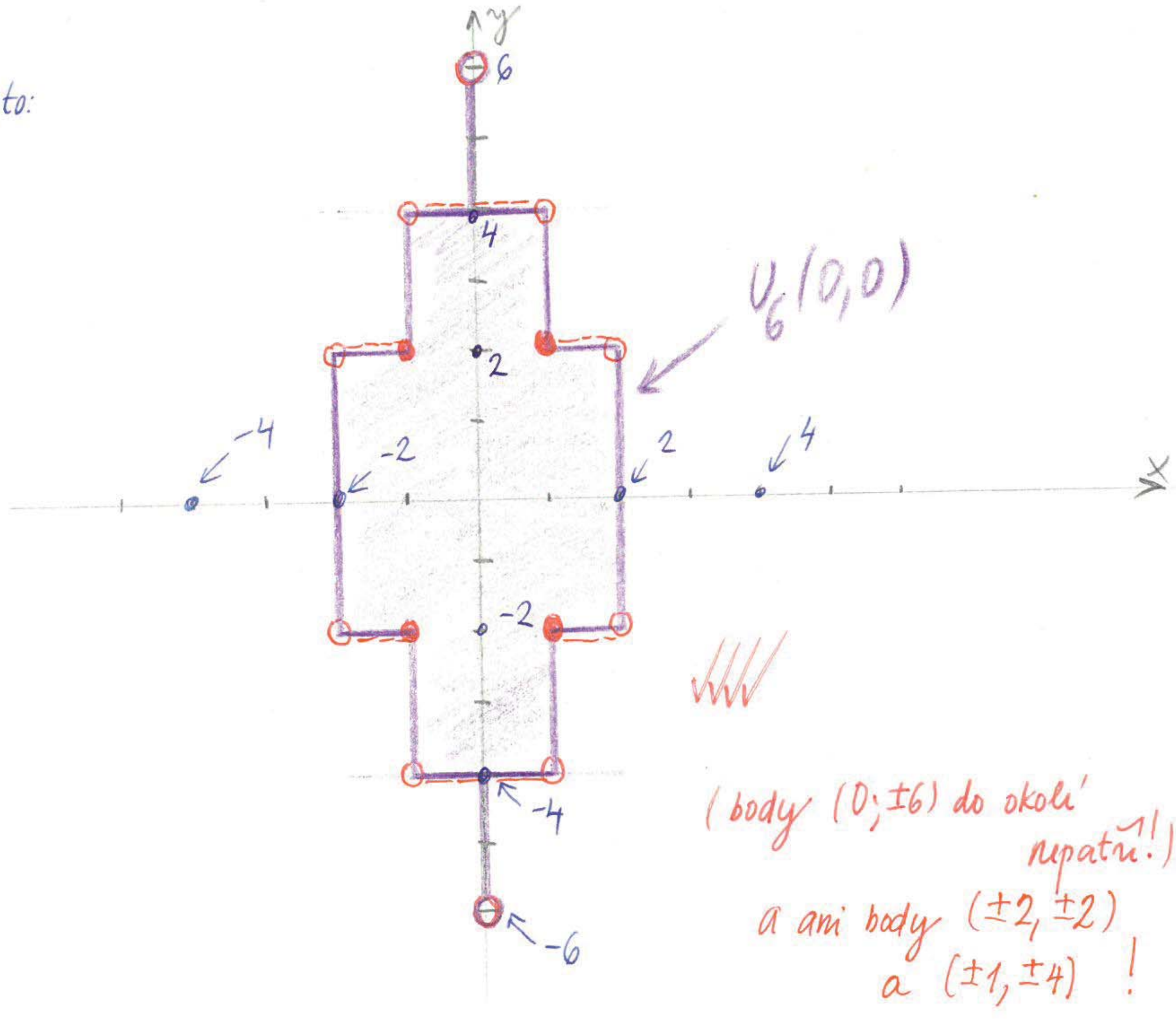
$$2 \cdot 1 + |x_2| < 6$$
$$|x_2| < 4$$

γ) if $|x_1| \in (1, 2\rangle$ then $\lceil |x_1| \rceil = 2$ a hledáme body, pro které

$$2 \cdot 2 + |x_2| < 6$$
$$|x_2| < 2$$

δ) if $|x_1| > 2$ then $\lceil |x_1| \rceil \geq 3$ a nerovnost $2\overbrace{\lceil |x_1| \rceil}^{=3} + |x_2| < 6$ nelze splnit žádnou volbou $x_2 \in \mathbb{R}$

Proto:



(body $(0; \pm 6)$ do okolí nepatří!)

a ani body $(\pm 2, \pm 2)$

a $(\pm 1, \pm 4)$!

$\langle g|g\rangle = 5 \qquad \langle f|f\rangle = 3 \qquad \rho(f,g) = 2 \qquad \rho(f;ig) = \sqrt{2}$

Pomocné výpočty:

$$\langle ig|ig\rangle = i\cdot i^{*}\langle g|g\rangle = |i|\langle g|g\rangle = \langle g|g\rangle$$

$$\langle ig|f\rangle^{*} = \left[i\langle g|f\rangle\right]^{*} = \left[i\,\mathrm{Re}\langle g|f\rangle - \mathrm{Im}\langle g|f\rangle\right]^{*} = -i\,\mathrm{Re}\langle g|f\rangle - \mathrm{Im}\langle g|f\rangle$$

Výpočet:

$$\rho^2(f,g) = \|f-g\|^2 = \langle f-g|f-g\rangle = \langle f|f\rangle + \langle g|g\rangle - \langle g|f\rangle - \langle f|g\rangle =$$

$$= \langle f|f\rangle + \langle g|g\rangle - \langle g|f\rangle - \langle g|f\rangle^{*} =$$

$$= \langle f|f\rangle + \langle g|g\rangle - 2\,\mathrm{Re}\langle g|f\rangle = 3 + 5 - 2\,\mathrm{Re}\langle g|f\rangle = 2^2$$

$$\underline{\mathrm{Re}\langle g|f\rangle = 2}$$

$$\rho^2(f,ig) = \|f-ig\|^2 = \langle f-ig|f-ig\rangle = \langle f|f\rangle + \langle g|g\rangle - \langle ig|f\rangle - \langle f|ig\rangle =$$

$$= \langle f|f\rangle + \langle g|g\rangle - i\langle g|f\rangle - \langle ig|f\rangle^{*} =$$

$$= \langle f|f\rangle + \langle g|g\rangle - i\,\mathrm{Re}\langle g|f\rangle + \mathrm{Im}\langle g|f\rangle + i\,\mathrm{Re}\langle g|f\rangle + \mathrm{Im}\langle g|f\rangle =$$

$$= \langle f|f\rangle + \langle g|g\rangle + 2\,\mathrm{Im}\langle g|f\rangle = 5 + 3 + 2\,\mathrm{Im}\langle g|f\rangle = 2$$

$$\underline{\mathrm{Im}\langle g|f\rangle = -3}$$

$$\underline{\langle g|f\rangle = 2 - 3i}$$

$$Q(x,y,z) = (x-2y+z)^2 - 4y^2 - z^2 + 4yz + 3y^2 - 6yz + 9z + 9 =$$
$$= (x-2y+z)^2 - y^2 - 2yz - z^2 + 9z + 9 =$$
$$= (x-2y+z)^2 - (y+z)^2 + 9z + 9$$

Substituce:

$$\left.\begin{aligned} \xi &= x-2y+z \\ \eta &= y+z \\ \lambda &= z \end{aligned}\right\} \Rightarrow \begin{aligned} x &= \xi + 2\eta - 2\lambda - \lambda = \xi + 2\eta - 3\lambda \\ y &= \eta - \lambda \\ z &= \lambda \end{aligned}$$

$$Q(\xi,\eta,\lambda) = \xi^2 - \eta^2 + 9\lambda + 9 = a^2 - b^2 - 2c$$

Klasifikace:

$$\left.\begin{aligned} &\text{název: } \overset{\text{hyperbolický}}{\cancel{\text{eliptický}}} \text{ paraboloid} \\ &sg(Q) = (1,1,1) \\ &SG(Q) = (2,2,0) \\ &\text{regulární \& necentrální} \end{aligned}\right\} \text{normální tvar: } a^2 - b^2 - 2c = 0$$

Normalizující transformace:

$$\left.\begin{aligned} \xi &= a \\ \eta &= b \\ -\tfrac{9}{2}\lambda - \tfrac{9}{2} &= c \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \xi &= a \\ \eta &= b \\ \lambda &= -\tfrac{2}{9}c - 1 \end{aligned}\right\} \Rightarrow \begin{aligned} x &= a + 2b + \tfrac{2}{3}c + 3 \\ y &= b + \tfrac{2}{9}c + 1 \\ z &= -\tfrac{2}{9}c - 1 \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2/3 \\ 0 & 1 & 2/9 \\ 0 & 0 & -2/9 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n \cdot x}{\sqrt{1+3n^4x^2}} \qquad \text{označme: } g_n(x) = \frac{x}{\sqrt{1+3n^4x^2}}$$

$$g_n'(x) = \frac{\sqrt{1+3n^4x^2} - x \dfrac{\frac{1}{2} 3n^4 \cdot 2x}{\sqrt{1+3n^4x^2}}}{1+3n^4x^2} = \frac{1+3n^4x^2 - 3n^4x^2}{(1+3n^4x^2)^{3/2}} = \frac{1}{(1+3n^4x^2)^{3/2}}$$

$g_n'(x) > 0$ na $\langle 0; +\infty)$ $\Rightarrow$ $g_n(x)$ je tam rostoucí

$$\Rightarrow \quad \sup g_n(x) = \lim_{x \to +\infty} g_n(x) = \frac{1}{\sqrt{3n^4}} = \frac{1}{\sqrt{3}} \cdot \frac{1}{n^2}$$

Odtud:

$$\left| (-1)^{n+1} \frac{nx}{\sqrt{1+3n^4x^2}} \right| \leq \frac{n}{\sqrt{3} \cdot n^2} = \frac{1}{\sqrt{3} \cdot n}$$

Řada $\sum_n \frac{1}{\sqrt{3}n}$ je ale divergentní $\Rightarrow$ Weierstraß nerozhodne

Abelovo kritérium

- členěmí: $f_n(x) = \frac{(-1)^{n+1}}{n}$ & $g_n(x) = \frac{n^2 x}{\sqrt{1+3n^2x^2}}$

$\alpha)$ $\sum f_n(x)$ SK na $\langle 0; +\infty)$ $\Leftarrow$ bodová = stejnoměrná

$\beta)$ monotonie $g_n(x)$:

$$g_{n+1}(x) > g_n(x)$$

$$(n+1)^4(1+3n^4x^2) > n^4(1+3(n+1)^4x^2)$$

$$(n+1)^4 + 3n^4x^2(n+1)^4 > n^4 + 3n^4x^2(n+1)^4$$

$$n+1 > n$$

$\gamma)$ omezenost (stejnoměrná) $g_n(x)$:

- funkce $\frac{x}{\sqrt{1+3n^4x^2}}$ je omezená funkcí $\frac{1}{\sqrt{3} \cdot n^2}$ (viz výše)

$$\Rightarrow \quad |g_n(x)| \leq n^2 \cdot \frac{1}{\sqrt{3}n^2} = \frac{1}{\sqrt{3}}$$

$\alpha)$ & $\beta)$ & $\gamma)$ $\Rightarrow$ řada konverguje podle Abelova kritéria na $\langle 0; +\infty)$ stejnoměrně ✓

$$y'' - 2\frac{y}{x^2} = 54 \ln(x)$$

Eulerova rovnice: $x = e^t$ & $t = \ln x$ & $\frac{dt}{dx} = \frac{1}{x}$

$$y' = \dot{y}\frac{1}{x} \quad \& \quad y'' = \ddot{y}\frac{1}{x^2} - \dot{y}\frac{1}{x^2}$$

$$\ddot{y} - \dot{y} - 2y = 54 e^{2t} \cdot t$$

$$\underbrace{\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)}$$

$$FS_{(t)} = \{e^{2t}; e^{-t}\}$$

Předpověď partikulárního řešení: $y_p(t) = (at + b) \cdot t \cdot e^{2t}$

$$\dot{y}_p = (2at + b + 2at^2 + 2bt)e^{2t}$$

$$\ddot{y}_p = (2a + 4at + 2b + 4at + 2b + 4at^2 + 4bt)e^{2t}$$

Dosazení a určení konstant a, b:

$$2a + 4at + 2b + 4at + 2b + \cancel{4at^2} + \cancel{4bt} - 2at - b - \cancel{2at^2} - \cancel{2bt} - \cancel{2at^2} - \cancel{2bt} \stackrel{!}{=} 54t$$

$$6at \stackrel{!}{=} 54t \quad \& \quad 2a + 3b \stackrel{!}{=} 0$$

$$a = 9 \quad \& \quad b = -6$$

$$y(t) = \alpha e^{2t} + \beta e^{-t} + (9t - 6) \cdot t \cdot e^{2t}$$

$$y(x) = \alpha x^2 + \frac{\beta}{x} + (9\ln x - 6) \cdot \ln x \cdot x^2$$

$$Dom(y) = (0; +\infty)$$

$$\varkappa(\vec{x},\vec{y}) = a|x_1 - y_1| + b|x_2 - y_2| \qquad a, b \in \mathbb{R}^+$$

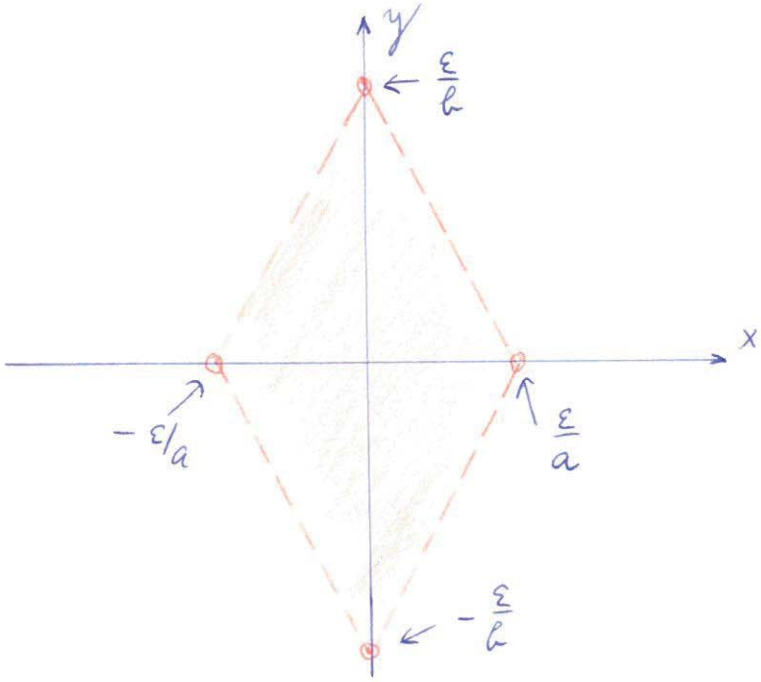
$$U_\varepsilon(\vec{0}) = \{\vec{x} \in \mathbb{R}^2 : \varkappa(\vec{x},\vec{0}) < \varepsilon\} = \{\vec{x} \in \mathbb{R}^2 : a|x_1| + b|x_2| < \varepsilon\}$$

Nerovnici $a|x_1| + b|x_2| < \varepsilon$ řešíme nejprve v 1. kvadrantu:

•) $ax + by < \varepsilon \Rightarrow by < \varepsilon - ax \Rightarrow y < \frac{\varepsilon}{b} - \frac{a}{b}x$

••) hraniční přímka má rovnici $y = \frac{\varepsilon}{b} - \frac{a}{b}x$

$$\Rightarrow y(0) = \frac{\varepsilon}{b} \;\&\; \left(y = 0 \Rightarrow x = \frac{\varepsilon}{a}\right)$$



Obsah:

$$\mu_2(U_\varepsilon(\vec{0})) = 4 \cdot \frac{1}{2} \cdot \frac{\varepsilon}{a} \cdot \frac{\varepsilon}{b} = 2\frac{\varepsilon^2}{ab}$$

$$g(x) = \int_3^x \frac{1}{\sqrt{y+1}}\,dy \qquad h(y) = (1+y)^{-1/2}$$

$$h'(y) = -\frac{1}{2}(1+y)^{-3/2} \qquad h''(y) = \frac{1}{2}\cdot\frac{3}{2}(1+y)^{-5/2}$$

$$h^{(n)}(y) = \frac{(2n-1)!!}{2^n}(-1)^n (1+y)^{-\frac{(2n+1)}{2}}$$

$$h^{n}(3) = (-1)^n \frac{(2n-1)!!}{2^n}\left(\frac{1}{2}\right)^{2n+1}$$

$$h(y) = \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(2n-1)!!}{2^n}\left(\frac{1}{2}\right)^{2n}\cdot(-1)^n\frac{1}{n!}(y-3)^n =$$

$$= \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n \frac{(2n-1)!!}{8^n}\frac{1}{n!}(y-3)^n$$

$$R^{-1} = \lim_{n\to\infty}\left|\frac{(2n+1)!!}{8^{n+1}}\cdot\frac{n!}{(n+1)!}\,\frac{8^n}{(2n-1)!!}\right| = \lim_{n\to\infty}\frac{2n+1}{8n+8} = \frac{1}{4}$$

$$R = 4 \quad\Rightarrow\quad I = (-1,7)$$

$$g(x) = \frac{1}{2}(x-3) + \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!!}{8^n\cdot n!}\int_3^x (y-3)^n\,dy =$$

$$= \text{—— " ——} \qquad \left[\frac{(y-3)^{n+1}}{n+1}\right]_3^x =$$

$$= \frac{1}{2}(x-3) + \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!!}{8^n\cdot n!}\,\frac{(x-3)^{n+1}}{(n+1)}$$

Integrováním se poloměr konvergence měnit nemůže $\Rightarrow R = 4$

Interval konvergence je tedy $I = (-1,7)$

Krajní body: $\displaystyle\sum_{n=1}^{\infty}(\pm 1)^n\frac{(2n-1)!!}{(2n)!!}\,\frac{1}{n+1} \quad\Rightarrow\quad O = \langle -1; 7\rangle$

Raabe

$$\lim_{n\to\infty} n\left(1 - \frac{(2n+1)!!}{(2n+2)!!}\,\frac{n+1}{n+2}\cdot\frac{(2n)!!}{(2n-1)!!}\right) = \lim_{n\to\infty} n\,\frac{2n^2+6n+4-2n^2-3n-1}{2n^2+6n+4} = \frac{3}{2} > 1$$

9b

```
Sum[(-1)^n x^(3 n) / Factorial[3 n], {n, 0, Infinity}] // FullSimplify
```

$$\frac{1}{3}\left(e^{-x}+2\,e^{x/2}\,\mathrm{Cos}\left[\frac{\sqrt{3}\,x}{2}\right]\right)$$

```
DSolve[{q'''[x] + q[x] == 0, q[0] == 1, q'[0] == 0, q''[0] == 0}, q[x], x] // FullSimplify
```

$$\left\{\left\{q[x]\to\frac{1}{3}\left(e^{-x}+2\,e^{x/2}\,\mathrm{Cos}\left[\frac{\sqrt{3}\,x}{2}\right]\right)\right\}\right\}$$

$$s(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{3n}}{(3n)!}\qquad s'''(x)=\sum_{n=1}^{\infty}(-1)^n\frac{x^{3n-3}}{(3n-3)!}=\left|\,m=n-1\,\right|=$$

$$=\sum_{m=0}^{\infty}(-1)^{m+1}\frac{x^{3m}}{(3m)!}=-s(x)$$

$$\left.\begin{array}{l} s(0)=1 \quad s'(0)=0 \quad s''(0)=0 \\ s'''+s=0 \end{array}\right\}\ \text{Cauchyova úloha}$$

$$\lambda^3+1=0 \qquad (\lambda^3+1):(\lambda+1)=\lambda^2-\lambda+1 \qquad \lambda_{2,3}=\frac{1}{2}\left(1\pm\sqrt{-3}\right)$$

$$-\lambda^2+1 \qquad\qquad \lambda_{2,3}=\frac{1}{2}\left(1+\sqrt{3}i\right)$$

$$\lambda+1$$

$$s(x)=\alpha e^{-x}+\beta e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)+\mu e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

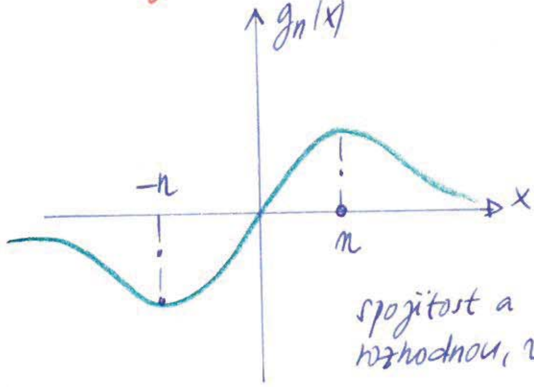
$$\left.\begin{array}{l} s(0)=1 \\ s'(0)=0 \\ s''(0)=0 \end{array}\right\}\Rightarrow(\alpha,\beta,\mu)=\left(\frac{1}{3};\frac{2}{3};0\right)$$

$$s(x)=\frac{1}{3}\left(e^{x}+2e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)\right);\ \mathrm{Dom}(s)=\mathbb{R}$$

$\mathrm{Dom}(s)=\mathbb{G}$

$$g_n(x) \stackrel{\Delta}{=} \frac{x}{n^2+x^2}$$

$$g_n'(x) = \frac{n^2+x^2-2x^2}{(n^2+x^2)} \stackrel{!}{=} 0 \quad \Rightarrow \quad x = \pm n$$



Žádné zdůvodnění, že stacionární bod je skutečně maximem

spojitost a limity v $\pm\infty$ rozhodnou, že

$$\sup_{x \in \mathbb{R}} |g_n(x)| = g_n(n) = \frac{n}{2n^2}$$

$\Rightarrow \forall x \in \mathbb{R}$:

$$\left| \frac{n^3 2^n x}{e^n(n^2+x^2)} \right| \leq \frac{n^3 \cdot 2^n}{e^n} \cdot \frac{1}{2 \cdot n} = \frac{n^2 \cdot 2^{n-1}}{e^n}$$

← korektní zápis BEZ SUM!

Číselná řada $\sum_n \frac{n^2 2^{n-1}}{e^n}$ konverguje z podílového kritéria:

$$\lim_{n\to\infty} \frac{(n+1)^2 \cdot 2^n}{e^{n+1}} \cdot \frac{e^n}{n^2 \cdot 2^{n-1}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{2}{e} = \underbrace{\frac{2}{e} < 1}_{\text{těsně :-)}}$$

$e \doteq 2{,}71\ldots$

$\Rightarrow$ Podle W-kritéria zadaná funkční řada konverguje stejnoměrně na $\mathbb{R}$

$\Omega_0 := \mathrm{Ker}\,\hat{L} = \{ y(x) \in C^2(I) : y'' + (2 - \frac{2}{x}) y' + \frac{2 + x(x-2)}{x^2} y = 0 \}$ 6b

$\tilde{\Omega}_0 := \mathrm{Ker}\,\hat{K} = \{ y(x) \in C^3(I) : y''' + 2y'' + y' = 0 \}$

$\Omega_0 \cap \tilde{\Omega}_0 = \{ y(x) \in C^2(I) : y''' + 2y'' + y' = 0 \wedge y'' + (2 - \frac{2}{x}) y' + \frac{2 + x(x-2)}{x^2} y = 0 \}$

$\Rightarrow$ Funkce z $\Omega_0 \cap \tilde{\Omega}_0$ musejí tedy řešit obě rovnice

•) snazší bude řešit rovnici $y''' + 2y'' + y' = 0$ a zkoumat, která z jejích řešení řeší i rovnici $\hat{L}(y(x)) = 0$

za pochopení zadání!

$$\lambda^3 + 2\lambda^2 + \lambda = \lambda(\lambda^2 + 2\lambda + 1) = \lambda(\lambda+1)^2 \overset{!}{=} 0$$

$$\tilde{\Omega}_0 = [1, e^{-x}, x \cdot e^{-x}]_\lambda$$

$$\hat{L}(1) = \frac{2 + x(x-2)}{x^2} \neq 0 \Rightarrow 1 \notin \Omega_0$$

$$\hat{L}(e^{-x}) = e^{-x}\left(1 - 2 + \frac{2}{x} + \frac{2}{x^2} + 1 - \frac{2}{x}\right) \neq 0 \Rightarrow e^{-x} \notin \Omega_0$$

$$\hat{L}(x \cdot e^{-x}) = \left| \begin{array}{l} z = xe^{-x} \\ z' = (1-x)e^{-x} \\ z'' = (-1-1+x)e^{-x} = (-2+x)e^{-x} \end{array} \right| =$$

$$= e^{-x}\left[-2 + x + (2 - \frac{2}{x})(1-x) + \frac{2}{x} + x - 2\right] =$$

$$= e^{-x}\left[-2 + x + 2 - 2x - \cancel{\frac{2}{x}} + 2 + \cancel{\frac{2}{x}} + x - 2\right] = 0 \Rightarrow xe^{-x} \in \Omega_0$$

$$\Rightarrow \quad \Omega_0 \cap \tilde{\Omega}_0 = [xe^{-x}]_\lambda$$

průnik jader operátorů $\hat{L}, \hat{K}$

minus 2 body!

Pozor na odfláknutý zápis: $\Omega_0 \cap \tilde{\Omega}_0 = \{xe^{-x}\}$

a na fuj-zápis: $\hat{K} \cap \hat{L} = xe^{-x}$

$$x^3 y''' - 2x^2 y'' - 8xy' + 20y = \left(\frac{28}{x}\right)^2$$

11

- $x > 0$ $\quad x = e^t \Leftrightarrow t = \ln(x)$

$$y' = \frac{1}{x}\dot{y}$$
$$y'' = -\frac{1}{x^2}\dot{y} + \frac{1}{x^2}\ddot{y}$$
$$y''' = \frac{2}{x^3}\dot{y} - \frac{3}{x^3}\ddot{y} + \frac{1}{x^3}\dddot{y}$$

$$2\dot{y} - 3\ddot{y} + \dddot{y} + 2\dot{y} - 2\ddot{y} - 8\dot{y} + 20y = 28^2 e^{-2t}$$

$$\dddot{y} - 5\ddot{y} - 4\dot{y} + 20y = 28^2 \cdot e^{-2t}$$

$$\lambda^3 - 5\lambda^2 - 4\lambda + 20 = \lambda^2(\lambda-5) - 4(\lambda-5) = (\lambda-5)(\lambda^2-4) = 0$$

$$F_S = \{e^{5t};\ e^{-2t};\ e^{2t}\}$$

$$L(y_p) = 28^2 \cdot e^{-2t} \qquad y_p(t) = a t e^{-2t}$$

$$\dot{y}_p = a(1-2t)e^{-2t}$$
$$\ddot{y}_p = 4a(t-1)e^{-2t}$$
$$\dddot{y}_p = 4a(\cancel{}\,{-2t} + 3)e^{-2t}$$

$$a[-8t + 12 - 20t + 20 - 4 + 8t + 20t]e^{-2t} = 28^2 e^{-2t}$$
$$28a = 28^2$$
$$a = 28$$

$$y(t) = C_1 e^{5t} + C_2 e^{-2t} + C_3 e^{2t} + 28t e^{-2t}$$

$$y(x) = C_1 x^5 + \frac{C_2}{x^2} + C_3 x^2 + \frac{28}{x^2}\ln(x)\ ; \quad x > 0$$

- $x < 0$ $\quad x = -e^t$

$$\dddot{y} - 5\ddot{y} - 4\dot{y} + 20y = 28^2 e^{-2t}$$
$$\Longrightarrow y(t) = C_1 e^{5t} + C_2 e^{-2t} + C_3 e^{2t} + 28t e^{-2t}$$

$$y(x) = C_1 x^5 + \frac{C_2}{x^2} + C_3 x^2 + \frac{28}{x^2}\ln(-x)\ ; \quad x < 0$$

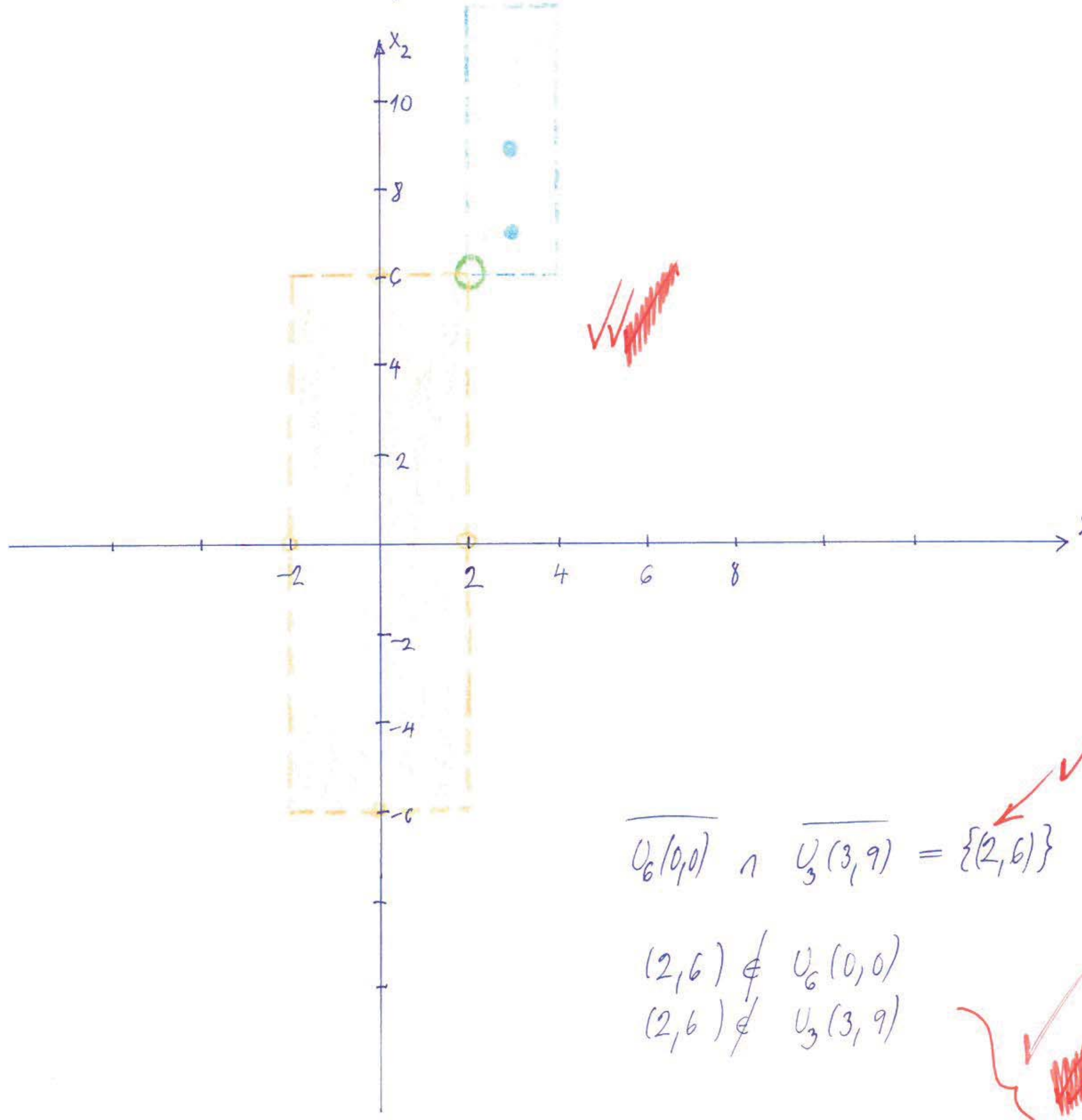
8l

Nechť je v prostoru $\mathbb{R}^2$ zadána metrika

$$\rho(\bar{x},\bar{y}) = \max\{3|x_1-y_1|;\ |x_2-y_2|\}.$$

Rozhodněte, zda existuje bod $\bar{a} \in \mathbb{R}^2$, který leží v okolí $U_6(0,0)$ i v okolí $U_3(3,9)$. Obě okolí načrtněte.

$$U_\varepsilon(\bar{a}) = \{\bar{y} \in \mathbb{R}^2 :\ \rho(\bar{y},\bar{a}) < \varepsilon\}$$



$$\overline{U_6(0,0)} \cap \overline{U_3(3,9)} = \{(2,6)\}$$

$$(2,6) \notin U_6(0,0)$$
$$(2,6) \notin U_3(3,9)$$

$\Rightarrow$ žádný takový bod neexistuje

$$U_6(0,0) \cap U_3(3,9) = \emptyset$$

Podezření: $\lim\limits_{n\to\infty} \vec{x_n} = \vec{a} = (1,1)$

$$\vec{x_n} - \vec{a} = \left(\frac{1}{n+2};\ \frac{n^2+6n+9-n^2-4n-4}{(n+2)^2}\right) = \left(\frac{1}{n+2};\ \frac{2n+5}{(n+2)^2}\right) \Rightarrow \rho(\vec{x_n},\vec{a}) = \max\left\{\frac{3}{n+2};\ \frac{2n+5}{(n+2)^2}\right\}$$

$\left[\text{platí ale, že } \frac{3}{n+2} > \frac{2n+5}{(n+2)^2} \Leftarrow 3n+6 < 2n+5\right] \Rightarrow \rho(\vec{x_n},\vec{a}) = \frac{3}{n+2}$

Pro libovolně malé $\varepsilon > 0$ lze vždy nalézt $n_0 \in \mathbb{N}$ tak, že: $n > n_0 \Rightarrow \frac{3}{n+2} < \varepsilon$

– stačí volit $n_0 := \lceil \frac{3}{\varepsilon} - 2 \rceil$ Proto $\lim\limits_{n\to\infty} \vec{x_n} = (1,1)$

důkaz

8b

In[1]:= **Expand[(x + μ y + μ u)^2 - μ^2 (y + u)^2 + (μ^2 - 9) u^2 - z^2]**

Out[1]= $-9u^2 + x^2 - z^2 + 2ux\mu + 2xy\mu + u^2\mu^2$

In[2]:= **u = 1**

Out[2]= 1

In[3]:= $-9u^2 + x^2 - z^2 + 2ux\mu + 2xy\mu + u^2\mu^2$

Out[3]= $-9 + x^2 - z^2 + 2x\mu + 2xy\mu + \mu^2$

In[4]:= **TeXForm[$-9 + x^2 - z^2 + 2x\mu + 2xy\mu + \mu^2$]**

Out[4]//TeXForm= `\mu ^2+x^2+2 \mu x+2 \mu x y-z^2-9`

$$Q(x,y,z) = x^2 - z^2 + 2\mu xy + 2\mu x + \mu^2 - 9$$

- je nutné přejít k rozšířené q-formě: $q_0(x,y,z,u) = x^2 - z^2 + 2\mu xy + 2\mu xu + (\mu^2-9)u^2$
- Lagrangeovým algoritmem:

$$q_0(x,y,z,u) = (x+\mu y+\mu u)^2 - \mu^2y^2 - \mu^2u^2 - 2\mu^2 yu + (\mu^2-9)u^2 - z^2 =$$
$$= (x+\mu y+\mu u)^2 - \mu^2(y+u)^2 - z^2 + (\mu^2-9)u^2$$

doplněn na čtverce

$\mu = 0 \Rightarrow SG(Q) = sg(q_0) = (1,2,1)$

$\mu = \pm 3 \Rightarrow SG(Q) = (1,2,1)$

$|\mu| \in (0,3) \Rightarrow SG(Q) = (1,3,0)$

$|\mu| > 3 \Rightarrow SG(Q) = (2,2,0)$

- výpočet středu:

$$A = \begin{pmatrix} 1 & \mu & 0 \\ \mu & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} -\mu \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & \mu & 0 \\ \mu & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \overset{!}{=} \begin{pmatrix} -\mu \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} s_1 + \mu s_2 \\ \mu s_1 \\ -s_3 \end{pmatrix} \overset{!}{=} \begin{pmatrix} -\mu \\ 0 \\ 0 \end{pmatrix} \Rightarrow s_3 = 0 \text{ vždy}$$

$s_1 = 0$ pokud $\mu \neq 0$, ale pro $\mu = 0$ může být $s_1 = \alpha$ $(\alpha \in \mathbb{R})$

$s_2 = -1$, pokud $\mu \neq 0$, ale pro $\mu = 0$ může být $s_2 = \beta$ $(\beta \in \mathbb{R})$, dle $\alpha \overset{!}{=} 0$

Závěr:

$\mu \neq 0 \Rightarrow \vec{s} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

$\mu = 0 \Rightarrow \vec{s} = \begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix} \quad \forall \beta \in \mathbb{R}$

```
In[1]:= y[x] = x;
        z[x] = x^4;

In[3]:= W = Det[{{D[y[x], {x, 1}], y[x]}, {D[z[x], {x, 1}], z[x]}}] // FullSimplify

Out[3]= -3 x^4

In[4]:= p = FullSimplify[
          Det[{{-D[y[x], {x, 2}], y[x]}, {-D[z[x], {x, 2}], z[x]}}] / W] // FullSimplify
```

Out[4]= $-\frac{4}{x}$

```
In[5]:= q = FullSimplify[
          Det[{{D[y[x], {x, 1}], -D[y[x], {x, 2}]}, {D[z[x], {x, 1}], -D[z[x], {x, 2}]}}] /
           W] // FullSimplify
```

Out[5]= $\frac{4}{x^2}$

```
DSolve[t''[x] + p * t'[x] + q * t[x] == 0, t[x], x] // FullSimplify
```

$\{\{t[x] \to e^{-x} x (C[1] + x C[2])\}\}$

$$\dot{y} = \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \left| x = e^t \right| = y' \cdot e^t$$

$$\ddot{y} = \frac{d}{dt}(y' \cdot e^t) = \frac{dy'}{dt} \cdot e^t + y' \cdot e^t = \frac{dy'}{dx} \cdot \frac{dx}{dt} \cdot e^t + y' \cdot e^t =$$

$$= y'' \cdot e^{2t} + y' e^{2t}$$

$$\Rightarrow \quad \dot{y} = y' \cdot x \quad \& \quad \ddot{y} = y'' x^2 + y' \cdot x$$

$$\ddot{y} - 5\dot{y} + 4y = y'' x^2 + y' x - 5y' x + 4y = y'' x^2 - 4y' x + 4y$$

$$x^2 y'' - 4x y' + 4y = 0$$

$$\int_0^\infty \sum_{n=1}^\infty x e^{-n^4x^2}\,dx = \left| \begin{matrix}\text{možnost záměny} \\ \text{prokážeme níže}\end{matrix} \right| = \sum_{n=1}^\infty \int_0^{+\infty} x e^{-n^4x^2}\,dx = \left| \begin{matrix} y = n^4x^2 \\ dy = 2n^4 x\,dx \end{matrix}\right| =$$

$$= \sum_{n=1}^\infty \int_0^{+\infty} \frac{1}{2n^4} e^{-y}\,dy = \sum_{n=1}^\infty \frac{1}{2n^4}\left[-e^{-y}\right]_0^\infty = \frac12 \sum_{n=1}^\infty \frac{1}{n^4} = \frac{\pi^4}{180}$$

Pro uskutečněnou záměnu je třeba, aby $\sum_n x e^{-n^4x^2} \overset{\langle 0;+\infty)}{\rightrightarrows}$

$$(xe^{-n^4x^2})' = (1-2x^2 n^4)\, e^{-n^4x^2} \overset{!}{=} 0$$

$$|x| = \frac{1}{\sqrt2}\cdot\frac{1}{n^2} \leftarrow \text{stac. bod}$$

$$g_n(x) \triangleq x e^{-n^4x^2} \Rightarrow g(0)=0 \wedge \lim_{x\to+\infty} g_n(x) = 0 \wedge g_n(x)\in C^1(\mathbb{R})$$

$\Rightarrow$ stac. bod $x_0 = \frac{1}{\sqrt2}\frac{1}{n^2}$ je bodem maxima

Proto: $$|xe^{-n^4x^2}| \leqq \frac{1}{\sqrt2}\frac{1}{n^2} e^{-1/2} \leqq \frac{1}{n^2}$$

Číselná řada $\sum_n 1/n^2$ stejnoměrně konverguje $\Rightarrow \sum g_n(x) \overset{\langle 0;+\infty)}{\rightrightarrows}$

Jenže použitá věta řeší záměny ve výraze $\int_a^b \sum_n g_n(x)$, kde $M=\langle a,b\rangle$.

$\Rightarrow$ víme tedy nyní, že $$\int_0^C \sum_{n=1}^\infty x e^{-n^4x^2}\,dx = \sum_{n=1}^\infty \int_0^C x e^{-n^4x^2}\,dx$$

Ještě potřebujeme, aby šlo zaměnit limitu a sumu pro řadu/výraz

$$\lim_{x\to+\infty}\sum_{n=1}^\infty \underbrace{\int_0^x y e^{-n^4y^2}\,dy}_{\triangleq h_n(x)}$$

- na to je potřeba, aby $h_n(x) \overset{(K;+\infty)}{\rightrightarrows}$

$$|h_n(x)| \leqq \int_0^{+\infty} y e^{-n^4y^2}\,dy = \frac{1}{2n^4} \quad \text{a} \quad \sum \frac{1}{2n^4} \text{ konverguje}$$

$\Rightarrow h_n(x) \overset{\mathbb{R}}{\rightrightarrows} \Rightarrow h_n(x) \overset{\langle K;+\infty)}{\rightrightarrows}$ Tedy OK

8b

$$R^{-1} = \lim_{n\to\infty} \frac{(4n+1)!!!!}{(n+1)!} \cdot \frac{n!}{(4n-3)!!!!} = \lim_{n\to\infty} \frac{4n+1}{n+1} = 4 \quad \Rightarrow \quad R = \frac{1}{4}$$

$$|x-1|^2 < \frac{1}{4}$$

$$|x-1| < \frac{1}{2}$$

Krajní body intervalu konvergence: $x_1 = \frac{1}{2}$; $x_2 = \frac{3}{2}$

Tvar řady v obou krajních bodech: $\sum_{n=1}^{\infty} \frac{(4n-3)!!!!}{n!} \left(-\frac{1}{4}\right)^n = \sum_{n=1}^{\infty} \frac{(4n-3)!!!!}{(4n)!!!!} (-1)^n$

nejsou-li krajní řady OK, dále se neopravuje!

Raabe:

$$\lim_{n\to\infty} n \left(1 - \frac{(4n+1)!!!!}{(4n+4)!!!!} \cdot \frac{(4n)!!!!}{(4n-3)!!!!}\right) = \lim_{n\to\infty} n \left(1 - \frac{4n+1}{4n+4}\right) =$$

$$= \lim_{n\to\infty} n \frac{3}{4n+4} = \frac{3}{4} \in (0,1)$$

$$\Rightarrow \quad O = \left\langle \frac{1}{2}; \frac{3}{2} \right\rangle$$

Kdyby bylo zadání $\sum_n \frac{(4n-3)!!!!}{n!} (x-1)^{2n}$, pak by $O = \left(\frac{1}{2}; \frac{3}{2}\right)$

- je třeba zkoumat řadu derivací, tj. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{-\frac{1}{2}\cdot 2x}{(x^2+8n^2)^{3/2}} =$

$$= \sum_{n=1}^{\infty} (-1)^{n} \frac{x}{(x^2+8n^2)^{3/2}} \quad (*)$$

- nejprve je ale nutno prověřit, zda původní (tj. zadaná) řada konverguje alespoň v jednom bodě — to ale platí např. pro $x=0$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{8}\, n} \quad (K)$$

- označme $h_n(x) = \dfrac{x}{(x^2+8n^2)^{3/2}}$ a zkoumejme její průběh

(nebo graf)

I. $h_n(0) = 0$

II. $h_n(x)$ lichá

III. $\lim_{x\to+\infty} h_n(x) = 0$

IV. $h_n(x)$ spojitá všude v $\mathbb{R}$, tj. $h_n(x) \in C(\mathbb{R})$

V. $$h_n'(x) = \frac{(x^2+8n^2)^{3/2} - x\cdot\frac{3}{2}\cdot 2x\cdot(x^2+8n^2)^{1/2}}{(x^2+8n^2)^3} = \frac{x^2+8n^2-3x^2}{(x^2+8n^2)^{5/2}}$$

$$h_n'(x) = 0 \iff 8n^2 - 2x^2 = 0 \Rightarrow x = \pm 2n$$

$$h_n(2n) = \frac{2n}{(4n^2+8n^2)^{3/2}} = \frac{2n}{12^{3/2} n^3} = \frac{2}{3^{3/2}\cdot 8\, n^2} = \frac{1}{4\cdot 3^{3/2} n^2}$$

I, II, III, IV & V $\Rightarrow$ $x = 2n$ je bodem maxima $\Rightarrow$

(zcela korektní zápis s absol. hodnotou a bez sum!)

$$\forall x\in\mathbb{R}: \left| (-1)^n \frac{x}{(x^2+8n^2)^{3/2}} \right| \le \frac{1}{4\cdot 3^{3/2} n^2} \le \frac{1}{n^2}$$

- ale řada $\sum \frac{1}{n^2}$ (K) $\Rightarrow$ podle W-kritéria konverguje řada $(*)$ stejnoměrně na $\mathbb{R}$, tj. také stejnoměrně na libovolné podmnožině $\mathbb{R}$

precizní formulace

$$2y' = \frac{y}{x} - 3 - \frac{9x}{3x+y} \qquad y\left(\tfrac{2}{3}\right) = -2$$

- jde o homogenní rovnici $\Rightarrow$ $y(x) = x \cdot z(x)$ $\Rightarrow$ $y' = z + xz'$

$$2z + 2xz' = z - 3 - \frac{9x}{3x + xz} = z - 3 - \frac{9}{3+z}$$

$$-2xz' = z + 3 + \frac{9}{3+z}$$

$$-2xz' = \frac{z^2+6z+18}{3+z}$$

$$\int \frac{2z+6}{z^2+6z+18}\, dz = -\int \frac{1}{x}\, dx$$

$$\ln|z^2+6z+18| = -\ln|x| + C$$

$$z^2+6z+18 = \frac{C}{x}$$

$$\frac{y^2}{x^2} + 6\frac{y}{x} + 18 = \frac{C}{x}$$

$$y^2 + 6xy + 18x^2 + Dx = 0 \qquad D = ?$$

$$4 + 6\tfrac{2}{3}\cdot(-2) + 18\cdot\tfrac{4}{9} + D\tfrac{2}{3} = 0$$

$$4 - 8 + 8 + \tfrac{2}{3}D \overset{!}{=} 0 \quad \Rightarrow \quad D = -6$$

$$y^2 + 6xy + 18x^2 - 6x = 0$$

$$(y+3x)^2 + 9x^2 - 6x = 0$$

$\Rightarrow$ jde ~~[illegible]~~ o elipsu

$$A = \begin{pmatrix} 18 & 3 \\ 3 & 1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 18 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad \Rightarrow \quad (s_1, s_2) = \left(\tfrac{1}{3}, -1\right)$$

(lineární řešení netřeba hledat)

9b

$$q(x_1, x_2, x_3) = \vec{x}^T \begin{pmatrix} \mu & -1 & -1 \\ -1 & \mu & -1 \\ -1 & -1 & \mu \end{pmatrix} \vec{x} \triangleq \vec{x}^T A \vec{x}$$

$$\underbrace{q(\vec{x}) \trianglelefteq 0 \ \vee \ q(\vec{x}) \trianglerighteq 0 \ \Rightarrow \ s_3 \neq 0 \ \left(sg(\vec{q}) = (s_1, s_2, s_3) \right) \ \Rightarrow \ \det A = 0}_{\checkmark}$$

nutná podmínka pro indefinitnost (ne postačující)

$\Rightarrow$ Hledejme ta $\mu \in \mathbb{R}$, pro něž $\det(A) = 0$

$$\det(A) = \mu^3 - 3\mu - 2 \ \checkmark \qquad (\mu^3 - 3\mu - 2) : (\mu + 1) = \mu^2 - \mu - 2$$

$$\mu_{2,3} = \frac{1}{2}(1 \pm 3) = \begin{cases} 2 \\ -1 \end{cases}$$

Pro $\mu \in \{-1, 2\}$ ✓✓ je zadaná q-forma ~~ind~~ semidefinitní nebo indefinitní

•) $\mu = -1 \ \Rightarrow \ A = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$ ✓ ← nějaká smysluplná cesta, jak zjistit spektrum

$$\det(A - \lambda \mathbb{I}) = \begin{vmatrix} -1-\lambda & -1 & -1 \\ -1 & -1-\lambda & -1 \\ -1 & -1 & -1-\lambda \end{vmatrix} = -\lambda^2 (3 + \lambda)$$

$$\vec{\sigma}(A) = (-3, 0, 0) \ \checkmark \quad \Rightarrow \quad q(\vec{x}) \trianglelefteq 0 \ \checkmark$$

•) $\mu = 2 \ \Rightarrow \ A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$

$$\det(A - \lambda \mathbb{I}) = \begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = -\lambda(-3 + \lambda)^2$$

$$\vec{\sigma}(A) = (3, 3, 0) \ \checkmark \quad \Rightarrow \quad q(\vec{x}) \trianglerighteq 0 \ \checkmark$$

Pozor na častou a hrubou chybu:

Sylvesterovo kritérium pro semidefinitnost nefunguje. Jeho aplikace vede na chybné výsledky!