$f, g \in \mathcal{H}$ $\qquad \alpha_n = \langle f | \varphi_n \rangle = \frac{1}{n}$ & $\beta_n = \langle g | \varphi_n \rangle = \frac{1}{n+1}$

$\hat{L}$ musí být operátorem s čistě bodovým spektrem. Pak totiž každá funkce $f(x) \in \mathcal{H}$ je zapsatelná do formátu Fourierovy řady, tj.:

$$f(x) = \sum_{n=1}^{\infty} \underbrace{\langle f | \varphi_n \rangle}_{\alpha_n} \varphi_n(x) \quad \& \quad g(x) = \sum_{n=1}^{\infty} \underbrace{\langle g | \varphi_n \rangle}_{\beta_n} \varphi_n(x)$$

$$\rho^2(f,g) = \| f(x) - g(x) \|^2 = \langle f-g | f-g \rangle = \langle f|f \rangle + \langle g|g \rangle - \langle g|f \rangle -$$
$$- \langle f|g \rangle = \|f\|^2 + \|g\|^2 - 2 \operatorname{Re} \langle f|g \rangle$$

$$\langle f|f \rangle = \left\langle \sum_{n=1}^{\infty} \alpha_n \varphi_n(x) \middle| \sum_{m=1}^{\infty} \alpha_m \varphi_m(x) \right\rangle = \left| \begin{array}{l} \text{OSČBS} \Rightarrow \text{hermitovský} \Rightarrow \\ \Rightarrow \text{spojitý} \end{array} \right| =$$
$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle \alpha_n \varphi_n(x) | \alpha_m \varphi_m(x) \rangle = \sum_{n=1}^{\infty} \alpha_n \cdot \alpha_n^* \underbrace{\|\varphi_n(x)\|^2}_{=1} = \sum_{n=1}^{\infty} |\alpha_n|^2$$

$$\langle g|g \rangle = \sum_{n=1}^{\infty} |\beta_n|^2$$

$$\langle f|g \rangle = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle \alpha_n \varphi_n(x) | \beta_m \varphi_m(x) \rangle = \sum_{n=1}^{\infty} \alpha_n \beta_n^*$$

$$\rho^2(f,g) = \sum_{n=1}^{\infty} |\alpha_n|^2 + \sum_{n=1}^{\infty} |\beta_n|^2 - 2 \sum_{n=1}^{\infty} \alpha_n \beta_n^* = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} -$$
$$- 2 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{\pi^2}{6} + \frac{\pi^2}{6} - 1 - \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) =$$
$$= \frac{\pi^2}{3} - 1 - \lim_{k \to \infty} \left(1 \cancel{- \frac{1}{2} + \frac{1}{2}} \cancel{- \frac{1}{3} + \ldots + \frac{1}{k}} - \frac{1}{k+1} \right) =$$
$$= \frac{\pi^2}{3} - 1 - \lim_{k \to \infty} \left(1 - \frac{1}{k+1} \right) = \frac{\pi^2}{3} - 2 > 0$$

$$\Rightarrow \rho(f,g) = \sqrt{\frac{\pi^2}{3} - 2}$$

$$\varphi(x) = \mu \int_0^x \frac{y^2}{x} \varphi(y)\,dy + 8 e^{\frac{\mu}{2}x^2}$$

$$K_1(x,y) = \frac{y^2}{x} \quad \Rightarrow \quad K_2(x,y) = \int_y^x K_1(x,s)\cdot K_1(s,y)\,ds = \int_y^x \frac{s^2}{x}\cdot\frac{y^2}{s}\,ds =$$

$$= \frac{y^2}{x}\left[\frac{s^2}{2}\right]_y^x = \frac{1}{2}\frac{y^2}{x}(x^2-y^2)$$

$$K_3(x,y) = \int_y^x \frac{s^2}{x}\cdot\frac{1}{2}\frac{y^2}{s}(s^2-y^2)\,ds = \frac{1}{2}\frac{y^2}{x}\int_y^x s(s^2-y^2)\,ds = \left| \begin{matrix} t = s^2-y^2 \\ dt = 2s\,ds \end{matrix} \right| =$$

$$= \frac{1}{4}\frac{y^2}{x}\int_0^{x^2-y^2} t\,dt = \frac{1}{8}\frac{y^2}{x}(x^2-y^2)^2$$

$$K_4(x,y) = \frac{1}{8}\frac{y^2}{x}\int_y^x s(s^2-y^2)^2\,ds = \frac{1}{16}\cdot\frac{1}{3}\frac{y^2}{x^2}(x^2-y^2)^3$$

$$K_n(x,y) = \frac{1}{2^{n-1}(n-1)!}\frac{y^2}{x}(x^2-y^2)^{n-1}$$

Důkaz:

$$K_{n+1}(x,y) = \frac{1}{2^{n-1}(n-1)!}\int_y^x \frac{s^2}{x}\cdot\frac{y^2}{s}(s^2-y^2)^{n-1}\,ds = \frac{1}{2^{n-1}(n-1)!}\frac{y^2}{x}\int_y^x s(s^2-y^2)^{n-1}\,ds =$$

$$= \left| \begin{matrix} t = s^2-y^2 \\ dt = 2s\,ds \end{matrix} \right| = \frac{1}{2^n(n-1)!}\frac{y^2}{x}\int_0^{x^2-y^2} t^{n-1}\,dt = \frac{1}{2^n n!}\frac{y^2}{x}(x^2-y^2)^n$$

q.e.d.

$$R(x,y|\mu) := \sum_{n=1}^{\infty}\mu^{n-1}K_n(x,y) = \frac{y^2}{x}\sum_{n=1}^{\infty}\frac{\mu^{n-1}}{2^{n-1}(n-1)!}(x^2-y^2)^{n-1} =$$

$$= \frac{y^2}{x}\sum_{m=0}^{\infty}\frac{\mu^m}{2^m m!}(x^2-y^2)^m = \frac{y^2}{x} e^{\frac{\mu}{2}(x^2-y^2)}$$

Řešení:

$$\varphi(x) = \mu\int_0^x R(x,y|\mu) f(y)\,dy + e^{\frac{\mu}{2}x^2} = 8\mu\int_0^x \frac{y^2}{x} e^{\frac{\mu}{2}x^2}\cdot e^{-\frac{\mu}{2}y^2} e^{\frac{\mu}{2}y^2}\,dy + 8e^{\frac{\mu}{2}x^2} =$$

$$= 8\mu\frac{1}{x} e^{\frac{\mu}{2}x^2}\int_0^x y^2\,dy + 8e^{\frac{\mu}{2}x^2} = 8e^{\frac{\mu}{2}x^2}\left(1+\frac{1}{3}x^2\right)$$

Otázka stojí takto: $f_\sigma \in \mathcal{D}_{reg} \wedge \lim\limits_{\sigma\to\pm\omega} f_\sigma \overset{\mathcal{D}'}{=} f \overset{?}{\Rightarrow} \lim\limits_{\sigma\to\pm\omega} f_\sigma' \overset{\mathcal{D}'}{=} f'$

Zkusme dokázať:

$$\left(\lim_{\sigma\to\pm\omega} f_\sigma' ; \varphi(x)\right) = \lim_{\sigma\to\pm\omega} \left(f_\sigma' ; \varphi(x)\right) = -\lim_{\sigma\to\pm\omega} \left(f_\sigma ; \varphi'(x)\right) =$$

$$= \left| \begin{array}{l} \text{protože} \\ f_\sigma \xrightarrow{\mathcal{D}'} f \end{array} \right| = -\left(f ; \varphi'(x)\right) = \left(f' ; \varphi(x)\right) \qquad \forall \varphi(x) \in \mathcal{D}(R)$$

$$\Rightarrow \lim_{\sigma\to\pm\omega} f_\sigma' = f'$$

Odpověď na otázku ze zadání je tedy

$$-\frac{2\frac{x}{2\sigma^2}}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}} \equiv -\frac{x}{\sqrt{2\pi}\,\sigma^3} e^{-\frac{x^2}{2\sigma^2}} \xrightarrow[\sigma\to 0_+]{\mathcal{D}'} \delta'$$

NEBO:

$$\lim_{\sigma\to 0_+} \int_R \frac{x}{\sqrt{2\pi}\,\sigma^3} e^{-\frac{x^2}{2\sigma^2}} \varphi(x)\, dx = \left| \begin{array}{l} y = \frac{x}{\sigma} \\ dy = \frac{1}{\sigma} dx \end{array} \right| =$$

$$= \lim_{\sigma\to 0_+} \int \frac{y\cdot\sigma}{\sqrt{2\pi}\,\sigma^3} e^{-\frac{y^2}{2}} \varphi(\sigma y)\, \sigma\, dy = \frac{1}{\sqrt{2\pi}} \lim_{\sigma\to 0_+} \int_R \frac{\varphi(\sigma y)}{\sigma} y\, e^{-\frac{y^2}{2}} dy =$$

$$= \left| \begin{array}{ll} u = \frac{1}{\sigma}\varphi(\sigma y) & v' = y\cdot e^{-\frac{y^2}{2}} \\ u' = \varphi'(\sigma y) & v = -e^{-\frac{y^2}{2}} \end{array} \right| = \frac{1}{\sqrt{2\pi}} \lim_{\sigma\to 0_+} \left\{ \underbrace{\left[-\frac{1}{\sigma} e^{-\frac{y^2}{2}} \varphi(\sigma y)\right]_R}_{=0} + \int_R e^{-\frac{y^2}{2}} \varphi'(\sigma y)\, dy \right\} =$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{\sigma\to 0_+} \int_R e^{-\frac{y^2}{2}} \varphi'(\sigma y)\, dy = \left| \text{proč lze zaměnit?} \right| =$$

$$= \frac{1}{\sqrt{2\pi}} \varphi'(0) \int_R e^{-\frac{y^2}{2}} dy = \frac{1}{\sqrt{2\pi}} \varphi'(0)\cdot\sqrt{2\pi} = \varphi'(0) = -\left(\delta' ; \varphi(x)\right)$$

q.e.d.

$$\int_0^\infty \frac{e^{-ax}}{x}\left(\cos^2(\beta x)+\sin^2(\beta x)-\cos^2(\gamma x)-\sin^2(\beta x)\right)dx=\int_0^\infty \frac{e^{-ax}}{x}\left(\cos(2\beta x)-\cos(2\gamma x)\right)dx$$

$$\mathcal{L}\left[\cos(2\beta x)-\cos(2\gamma x)\right]=\frac{p}{p^2+4\beta^2}-\frac{p}{p^2+4\gamma^2}$$

$$\mathcal{L}\left[\frac{\cos(2\beta x)-\cos(2\gamma x)}{x}\right]=\int_p^\infty\left(\frac{q}{q^2+4\beta^2}-\frac{q}{q^2+4\gamma^2}\right)dq=$$

$$=\int_p^\infty\left(\frac{1}{4\beta^2}\frac{q}{1+\left(\frac{q}{2\beta}\right)^2}-\frac{1}{4\gamma^2}\frac{q}{1+\left(\frac{q}{2\beta}\right)^2}\right)dq=$$

$$=\left[\frac{1}{2}\ln\left(1+\frac{q^2}{4\beta^2}\right)-\frac{1}{2}\ln\left(1+\frac{q^2}{4\gamma^2}\right)\right]_p^\infty=\frac{1}{2}\left[\ln\frac{1+\frac{q^2}{4\beta^2}}{1+\frac{q^2}{4\gamma^2}}\right]_0^\infty=$$

$$=-\frac{1}{2}\ln\frac{1+\frac{p^2}{4\beta^2}}{1+\frac{p^2}{4\gamma^2}}$$

$$\mathcal{L}\left[\frac{e^{-ax}}{x}\left(\cos(2\beta x)-\cos(2\gamma x)\right)\right]=\frac{1}{2}\ln\frac{1+\frac{(p+a)^2}{4\beta^2}}{1+\frac{(p+a)^2}{4\gamma^2}}$$

$$I=\lim_{p\to 0_+}\mathcal{L}\left[f(x|a,\beta,\gamma)\right]=\lim_{p\to 0_+}\frac{1}{2}\ln\frac{1+\frac{(p+a)^2}{4\beta^2}}{1+\frac{(p+a)^2}{4\gamma^2}}=$$

$$=\frac{1}{2}\ln\frac{4\beta^2+a^2}{4\beta^2}\cdot\frac{4\gamma^2}{4\gamma^2+a^2}=\ln\sqrt{\frac{\gamma^2}{\beta^2}\frac{4\beta^2+a^2}{4\gamma^2+a^2}}=$$

$$=\underline{\ln\frac{\gamma}{\beta}\sqrt{\frac{4\beta^2+a^2}{4\gamma^2+a^2}}}$$

$f(x) \in \mathcal{L}_2(G) \wedge H \subset G \wedge \mu(H) < +\infty \Rightarrow f(\vec{x}) \in \mathcal{L}_1(H)$

a) uvažme $f(x), g(x) \in \mathcal{L}_2(G)$

$$\underbrace{2|f(x)g^*(x)| \leq |f(x)|^2 + |g(x)|^2 \in \mathcal{L}(G)}_{\text{srovnávací kritérium}}$$

$$\Downarrow$$

$$|f(x)g^*(x)| \in \mathcal{L}(G) \wedge f(x)g^*(x) \in \mathcal{L}(G) \wedge$$

$$\wedge f(x)g^*(x) \in \mathcal{L}_1(G)$$

(banální obměny)

b) v předešlé úvaze položme:

$$g(\vec{x}) = \begin{cases} 1 & \vec{x} \in H \\ 0 & \vec{x} \in G \setminus H \end{cases}$$

- vidíme, že $g(\vec{x}) = |g(\vec{x})| = g^2(\vec{x})$
- navíc

$$\int_G g(\vec{x})\, d\vec{x} = \int_H g(\vec{x})\, d\vec{x} = \int_H 1\, d\vec{x} = \mu(H) \in \mathbb{R}$$

– lze tedy tvrdit, že skutečně $g(\vec{x}) \in \mathcal{L}_2(G)$

na H tedy platí:

$$2|f(\vec{x})g^*(\vec{x})| \leq |f(\vec{x})|^2 + 1$$

pokud je $f(\vec{x}) \in \mathcal{L}_2(G)$, je zjevně $f(\vec{x}) \in \mathcal{L}_2(H)$,

a tudíž $|f(\vec{x})|^2 + 1 \in \mathcal{L}(H)$

$\Rightarrow$ $|f(\vec{x})|^2 + 1$ je integrabilní majoranta a ze srovnávacího kritéria také $|f(\vec{x})g^*(\vec{x})| \in \mathcal{L}(H)$

$\Rightarrow$ protože ale na H: $f(\vec{x})g^*(\vec{x}) = f(\vec{x})$, plyne odtud, že

$$|f(\vec{x})| \in \mathcal{L}(H) \Leftrightarrow f(\vec{x}) \in \mathcal{L}_1(H)$$

a) v $\mathcal{S}(\mathbb{R})$

- předpokládejme, že pro nějaké $n \in \mathbb{N}_0$ platí: $\mathcal{F}[(ix)^n f(x)] = \frac{d^n}{d\xi^n} \mathcal{F}[f(x)]$
- dokažme, že pak platí podobný výraz také pro $n+1$

$$\frac{d^{n+1}F}{d\xi^{n+1}} = \frac{d}{d\xi} \mathcal{F}[(ix)^n f(x)] \overset{\mathcal{S}}{=} \frac{d}{d\xi} \int_{\mathbb{R}} (ix)^n f(x) e^{i\xi x} dx =$$

$= \Big|$ •) integrand je měřitelnou funkcí $\Leftarrow$ $\underbrace{(ix)^n e^{i\xi x} \in C(\mathbb{R}) \wedge f(x) \in \mathcal{S}}_{\Rightarrow \text{ integrand spojitý na } \mathbb{R}}$

•) pro $\xi = 0$: $(ix)^n f(x) \in \mathcal{L}(\mathbb{R}) \Leftarrow (ix)^n f(x) \in \mathcal{S}(\mathbb{R})$

•) $|(ix)^{n+1} f(x) e^{i\xi x}| \leq x^{n+1} f(x) \in \mathcal{L}(\mathbb{R}) \Leftarrow x^{n+1} f(x) \in \mathcal{S}$ $\Big| =$

$= \Big|$ využili jsme přitom inkluze $\mathcal{S} \subset \mathcal{L}$ $\Big|$ $= \int_{\mathbb{R}} (ix)^{n+1} f(x) e^{i\xi x} dx =$

$= \mathcal{F}[(ix)^{n+1} f(x)]$ q.e.d.

b) v $\mathcal{S}'(\mathbb{R})$

$$\left(\mathcal{F}\left[\frac{d^n f}{dx^n}\right]; \varphi(\xi)\right) = \left(\frac{d^n f}{dx^n}; \mathcal{F}[\varphi(\xi)](x)\right) = (-1)^n \left(f(x); \frac{d^n F}{dx^n}\right) =$$

$$= (-1)^n \left(f(x); \mathcal{F}[(i\xi)^n \varphi(\xi)]\right) = (-1)^n \left(\mathcal{F}[f(x)]; (i\xi)^n \varphi(\xi)\right) =$$

$$= (-1)^n \left((i\xi)^n \mathcal{F}[f(x)]; \varphi(\xi)\right) = \left((-i\xi)^n \mathcal{F}[f(x)](\xi); \varphi(\xi)\right)$$

$$\Rightarrow \quad \mathcal{F}\left[\frac{d^n f}{dx^n}\right] \overset{\mathcal{S}'}{=} (-i\xi)^n \mathcal{F}[f(x)]$$

$\hat{L}$ spojitý

vlastní funkce $\hat{L}$ nechť generují bázi v $\mathcal{H}$

$$\langle f | \varphi_n \rangle = \alpha_n$$

$$\langle g | \varphi_n \rangle = \beta_n$$

$$\hat{L} \varphi_n = \lambda_n \varphi_n$$

všechna vlastní čísla nechť leží na jednotkové kružnici, tj. $\lambda_n = e^{i \varkappa_n}$

– pak totiž

$$\lambda_n \in \mathbb{C} \quad \& \quad |\lambda_n| = |e^{i\varkappa_n}| = 1$$

$$\langle f | f \rangle = \Big\langle \sum_{n=1}^{\infty} \alpha_n \varphi_n \Big| \sum_{m=1}^{\infty} \alpha_m \varphi_m \Big\rangle = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle \alpha_n \varphi_n | \alpha_m \varphi_m \rangle =$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha_n \alpha_m^* \langle \varphi_n | \varphi_m \rangle = \sum_{n=1}^{\infty} |\alpha_n|^2$$

$$\langle g | g \rangle = \sum_{n=1}^{\infty} |\beta_n|^2$$

$$\langle f | g \rangle = \sum_{n=1}^{\infty} \alpha_n \beta_n^*$$

$$\langle \hat{L} f | \hat{L} f \rangle = \Big\langle \hat{L} \sum_{n=1}^{\infty} \alpha_n \varphi_n \Big| \hat{L} \sum_{m=1}^{\infty} \alpha_m \varphi_m \Big\rangle \overset{\hat{L}\text{ spojitý}}{=} \Big\langle \sum_{n=1}^{\infty} \hat{L}(\alpha_n \varphi_n) \Big| \sum_{m=1}^{\infty} \hat{L}(\alpha_m \varphi_m) \Big\rangle =$$

$$\underset{\text{spojitost sk. souč.}}{=} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle \hat{L}(\alpha_n \varphi_n) | \hat{L}(\alpha_m \varphi_m) \rangle \underset{\text{linearita}}{=} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle \alpha_n \hat{L}(\varphi_n) | \alpha_m \hat{L}(\varphi_m) \rangle =$$

$$\underset{\text{vlastní funkce}}{=} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha_n \alpha_m^* \langle \lambda_n \varphi_n | \lambda_m \varphi_m \rangle = \sum_{n=1}^{\infty} |\alpha_n|^2 \lambda_n \lambda_n^* =$$

$$= \sum_{n=1}^{\infty} |\alpha_n|^2 e^{i\varkappa_n} \cdot e^{-i\varkappa_n} = \sum_{n=1}^{\infty} |\alpha_n|^2$$

$$\langle \hat{L} g | \hat{L} g \rangle = \sum_{n=1}^{\infty} |\beta_n|^2$$

$$\langle \hat{L} f | \hat{L} g \rangle = \sum_{n=1}^{\infty} \alpha_n \beta_n^*$$

$$\rho^2(f,g) = \|f - g\|^2 = \langle f - g | f - g \rangle = \langle f | f \rangle + \langle g | g \rangle - \langle f | g \rangle - \langle f | g \rangle^* =$$

$$= \langle f | f \rangle + \langle g | g \rangle - 2 \operatorname{Im} \langle f | g \rangle = \sum_{n=1}^{\infty} |\alpha_n|^2 + \sum_{n=1}^{\infty} |\beta_n|^2 - 2 \operatorname{Im} \sum_{n=1}^{\infty} \alpha_n \beta_n^* =$$

$$= \langle \hat{L} f | \hat{L} f \rangle + \langle \hat{L} g | \hat{L} g \rangle - 2 \operatorname{Im} \langle \hat{L} f | \hat{L} g \rangle = \| \hat{L} f - \hat{L} g \| = \rho^2(\hat{L} f | \hat{L} g)$$

$\Rightarrow$ pro takový operátor se vzdálenost vzorů a obrazů nemění!

$$\lim_{n\to\infty} n\cdot e^{-n^2x^2}\cos(n\beta x) \quad \text{v} \quad \mathcal{D}'(\mathbb{R}) \qquad \beta\in\mathbb{R}$$

$$\lim_{n\to\infty}\left(\Theta(x)\cdot n\cdot e^{-n^2x^2}\cos(n\beta x);\ \varphi(x)\right) = \left|\begin{array}{l}\text{zdůvodnění, že } g(x)\in\mathcal{L}_{loc}(\mathbb{R}),\\ \text{a proto } \tilde{f}_n\in\mathcal{D}'_{reg}\end{array}\right| =$$

$$= \lim_{n\to\infty}\int_0^\infty n\, e^{-n^2x^2}\cos(n\beta x)\,\varphi(x)\,dx = \left|\begin{array}{l} nx = y\\ dx = \frac{1}{n}dy\end{array}\right| =$$

$$= \lim_{n\to\infty}\int_0^\infty e^{-y^2}\cos(\beta y)\,\varphi\!\left(\frac{y}{n}\right)dy = \left|\begin{array}{l}\text{zdůvodnění možnosti záměny}\\ \left|e^{-y^2}\cos(\beta y)\varphi\!\left(\frac{y}{n}\right)\right| \le e^{-y^2} \in\mathcal{L}(0,+\infty)\end{array}\right|$$

$$= \varphi(0)\int_0^\infty e^{-y^2}\cos(\beta y)\,dy =: \varphi(0)\,\Phi(\beta)$$

$$\frac{d\Phi}{d\beta} = \left|\begin{array}{l}\Phi(0) = \int_0^\infty e^{-y^2}dy = \frac{1}{2}\sqrt{\pi}\\ \left|e^{-y^2}y\cdot(-\sin\beta y)\right| \le y e^{-y^2}\in\mathcal{L}(0,+\infty) + \text{why}\end{array}\right| =$$

$$= -\int_0^\infty y e^{-y^2}\sin(\beta y)\,dy = \left|\begin{array}{ll} u = \sin(\beta y) & v' = y e^{-y^2}\\ u' = \beta\cos(\beta y) & v = -\frac{1}{2}e^{-y^2}\end{array}\right| =$$

$$= \left[+\frac{1}{2}\sin(\beta y)\,e^{-y^2}\right]_0^\infty - \frac{\beta}{2}\int_0^\infty \cos(\beta y)\,e^{-y^2}dy = -\frac{\beta}{2}\Phi(\beta)$$

$$\frac{d\Phi}{d\beta} + \frac{\beta}{2}\Phi(\beta) = 0 \quad \Big/ \cdot \text{I.F. } e^{+\frac{\beta^2}{4}}$$

$$\left(\Phi\cdot e^{+\frac{\beta^2}{4}}\right)' = 0 \quad\Rightarrow\quad \Phi(\beta) = C\cdot e^{-\frac{\beta^2}{4}} \ \wedge\ \Phi(0) = \frac{1}{2}\sqrt{\frac{\pi}{1}}$$

$$C = \frac{1}{2}\sqrt{\pi}$$

$$\Rightarrow \quad \lim_{n\to\infty}\tilde{f}_n = \frac{\sqrt{\pi}}{2}\delta(x)$$

$d(x,y) = 64y^2 - 4\cdot 8\cdot 2y^2 = 0 \Rightarrow$ rovnice je parabolická všude v R^2, což se dalo očekávat, je-li jeden vztah zadán

$\lambda(x,y) = +\frac{8y}{16} \qquad y' = -\frac{y}{2} \Rightarrow 2\ln y = -x + C \Rightarrow y^2 = K\cdot e^{-x}$

$$\boxed{\begin{aligned}\xi &= ye^x\\ \eta &= y^2e^x\end{aligned}} \quad \det\left(\frac{D(\xi,\eta)}{D(x,y)}\right) = \begin{vmatrix} ye^x & e^x \\ y^2e^x & 2ye^x \end{vmatrix} = y^2e^x$$

$$M_{reg} = \{(x,y)\in R^2 : y \neq 0\}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} ye^x + \frac{\partial u}{\partial \eta} y^2e^x \quad \& \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi} e^x + 2ye^x\frac{\partial u}{\partial \eta}$$

$$8\cdot \Big/ \quad \frac{\partial^2 u}{\partial x^2} = y^2e^{2x}\frac{\partial^2 u}{\partial \xi^2} + y^4e^{2x}\frac{\partial^2 u}{\partial \eta^2} + 2y^3e^{2x}\frac{\partial^2 u}{\partial \xi\partial \eta} + e^x y\frac{\partial u}{\partial \xi} + y^2e^x\frac{\partial u}{\partial \eta}$$

$$2y^2 \Big/ \quad \frac{\partial^2 u}{\partial y^2} = e^{2x}\frac{\partial^2 u}{\partial \xi^2} + 4y^2e^{2x}\frac{\partial^2 u}{\partial \eta^2} + 4ye^{2x}\frac{\partial^2 u}{\partial \xi\partial \eta} + \qquad + 2e^x\frac{\partial u}{\partial \eta}$$

$$-8y \Big/ \quad \frac{\partial^2 u}{\partial x\partial y} = ye^{2x}\frac{\partial^2 u}{\partial \xi^2} + 2y^3e^{2x}\frac{\partial^2 u}{\partial \eta^2} + (2y^2e^{2x} + y^2e^{2x})\frac{\partial^2 u}{\partial \xi\partial \eta} + e^x\frac{\partial u}{\partial \xi} + 2ye^x\frac{\partial u}{\partial \eta}$$

Dosazení:

$$2y^2e^{2x}\frac{\partial^2 u}{\partial \xi^2} - ye^x\frac{\partial u}{\partial \xi} = 8y^6e^{4x}$$

$$2\xi^2\frac{\partial^2 u}{\partial \xi^2} - \xi\frac{\partial u}{\partial \xi} = 8\xi^2\eta^2$$

$$\frac{\partial^2 u}{\partial \xi^2} - \frac{1}{2\xi}\frac{\partial u}{\partial \xi} = 4\eta^2 \qquad v(\xi,\eta) := \frac{\partial u}{\partial \xi}$$

$$\frac{\partial v}{\partial \xi} - \frac{1}{2\xi}v = 4\eta^2 \quad \Big/ \text{ I.F. } \frac{1}{\sqrt{\xi}}$$

$$\frac{d}{d\xi}\left(\frac{v}{\sqrt{\xi}}\right) = \frac{4\eta^2}{\sqrt{\xi}}$$

$$v(\xi,\eta) = 8\eta^2\sqrt{\xi}\cdot\sqrt{\xi} + C(\eta)\sqrt{\xi}$$

$$u(\xi,\eta) = 4\eta^2\xi^2 + D(\eta)\xi^{3/2} + E(\eta)$$

$$u(x,y) = 4y^6e^{4x} + D(y^2e^x)\cdot(ye^x)^{3/2} + E(y^2e^x)$$

$$x^n f(x) = 0 \quad / \ \mathcal{F}$$

$$\mathcal{F}[x^n f(x)] = 0 \qquad \mathcal{F}[f] = F(\xi)$$

$$(-i)^n \frac{d^n F}{d\xi^n} = 0$$

$$\underbrace{\frac{d^n F}{d\xi^n} = 0}_{\text{rovnice s konstantními koeficienty}}$$

$\lambda^n = 0$ charakteristický polynom

$\Rightarrow$ má n nulových kořenů

$\Downarrow$

$$F_S = \{1, \xi, \xi^2, \dots, \xi^{n-1}\}$$

$\Downarrow$

$$F(\xi) = C_1 + C_2 \xi + C_3 \xi^2 + \dots + C_n \xi^{n-1}$$

– hledejme vzory

$$\mathcal{F}[\delta(x)] = 1$$
$$\mathcal{F}[\delta'(x)] = (-i\xi)\,\mathcal{F}[\delta] = -i\xi$$
$$\mathcal{F}[\delta''(x)] = (-i\xi)(-i\xi) = -\xi^2$$

$\Downarrow$

$$\mathcal{F}[\delta^{(n)}(x)] = (-i)^n \xi^n \quad / \ \mathcal{F}$$
$$\mathcal{F}\mathcal{F}[\delta^{(n)}(x)] = (-i)^n \mathcal{F}[\xi^n]$$

$$f(x) = D_1 \delta(x) + D_2 \delta'(x) + D_3 \delta''(x) + \dots + D_n \delta^{(n-1)}(x)$$

$$y''' + 2y'' - 4y' - 8y = -12 - 24x \qquad y(0) = 2 \;\&\; y'(0) = 6 \;\&\; y''(0) = 12$$

$$\mathcal{L}[y] = Y \;\Rightarrow\; \mathcal{L}[y'] = pY - 2 \;\Rightarrow\; \mathcal{L}[y''] = p^2Y - 2p - 6 \;\Rightarrow$$

$$\Rightarrow\; \mathcal{L}[y'''] = p^3Y - 2p^2 - 6p - 12$$

$$\mathcal{L}[-12] = -\frac{12}{p} \quad \& \quad \mathcal{L}[-24x] = -\frac{24}{p^2}$$

Dosazení:

$$p^3Y - 2p^2 - 6p - 12 + 2p^2Y - 4p - 12 - 4pY + 8 - 8Y = -\frac{12}{p} - \frac{24}{p^2}$$

$$Y(p^3 + 2p^2 - 4p - 8) - 2p^2 - 10p - 16 = -\frac{24+12p}{p^2}$$

$$Y\cancel{(p^3+2p^2-4p-8)} = \frac{2p^4 + 10p^3 + 16p^2 - 12p - 24}{(p^3+2p^2-4p-8)p^2}$$

$$Y(p) = \frac{2p^4 + 10p^3 + 16p^2 - 12p - 24}{p^2[p(p^2-4) + 2(p^2-4)]} = \frac{2p^4 + 10p^3 + 16p^2 - 12p - 24}{p^2(p+2)(p+2)(p-2)}$$

$$Y(p) = \frac{A}{p} + \frac{B}{p^2} + \frac{C}{p+2} + \frac{D}{(p+2)^2} + \frac{E}{p-2}$$

$$(A, B, C, D, E) = (0, 3, 0, -1, 2)$$

numericky poněkud náchylnější na chyby

$$y(x) = \mathcal{L}^{-1}\left[\frac{3}{p^2}\right] - \mathcal{L}^{-1}\left[\frac{1}{(p+2)^2}\right] + \mathcal{L}^{-1}\left[\frac{2}{p-2}\right] =$$

$$= 3x \cdot \Theta(x) - \Theta(x) \cdot x \cdot e^{-2x} + 2\,\Theta(x)\,e^{2x}$$

1. Existence konvoluce

$$f(\vec{x}), g(\vec{x}) \in \mathcal{L}_1(\mathbb{E}^n)$$

$$\int_{\mathbb{E}^n}\int_{\mathbb{E}^n} |f(\vec{s}) \cdot g(\vec{x}-\vec{s})| \, d\vec{s} \, d\vec{x} = \int_{\mathbb{E}^n}\left(\int_{\mathbb{E}^n} |g(\vec{x}-\vec{s})| \, |f(\vec{s})| \, d\vec{x}\right) d\vec{s} =$$

$$= \int_{\mathbb{E}^n} |f(\vec{s})| d\vec{s} \cdot \int_{\mathbb{E}^n} |g(\vec{x}-\vec{s})| \, d\vec{x} = \underbrace{\int_{\mathbb{E}^n} |f(\vec{s})| \, d\vec{s}}_{<+\infty} \cdot \underbrace{\int_{\mathbb{E}^n} |g(\vec{y})| \, d\vec{y}}_{<+\infty}$$

$$f(\vec{s}) \in \mathcal{L}_1(\mathbb{E}^n) \qquad g(\vec{x}) \in \mathcal{L}_1(\mathbb{E}^n)$$

– z Fubiniovy věty odtud ale vyplývá, že $f(\vec{s})g(\vec{x}-\vec{s}) \in \mathcal{L}_1(\mathbb{E}^n)$ pro skoro všechna $\vec{x} \in \mathbb{E}^n$

2. Vztah $\mathcal{L}_1$-norem

$$\int_{\mathbb{E}^n} |f * g| \, d\vec{x} = \int_{\mathbb{E}^n} \left| \int_{\mathbb{E}^n} f(\vec{s}) g(\vec{x}-\vec{s}) \, d\vec{s} \right| d\vec{x} \leq$$

$$\leq \int_{\mathbb{E}^n}\int_{\mathbb{E}^n} |f(\vec{s})| \cdot |g(\vec{x}-\vec{s})| \, d\vec{s} \, d\vec{x} = \int_{\mathbb{E}^n} |f(\vec{s})| \, d\vec{s} \cdot \int_{\mathbb{E}^n} |g(\vec{x}-\vec{s})| \, d\vec{x} =$$

$$= \int_{\mathbb{E}^n} |f(\vec{s})| \, d\vec{s} \cdot \int_{\mathbb{E}^n} |g(\vec{y})| \, d\vec{y} = \|f\| \cdot \|g\|$$

3. Rovnost nastává, pokud $\operatorname{Ran}(f) \subset \langle 0; +\infty) \wedge \operatorname{Ran}(g) \subset \langle 0; +\infty)$

(spg: – nové zadání)

Předp.: $\hat{L}, \hat{K} : \mathcal{H} \mapsto \mathcal{H}$ omezené & OSČBS

- označme zmíněný společný systém vlastních funkcí $B = \{u_1(\vec{x}); u_2(\vec{x}); \dots\}$
- tento systém lze jistě ortonormalizovat, tj B lze převést na

$$\tilde{B} = \{\varphi_1(\vec{x}); \varphi_2(\vec{x}); \dots\}$$

máme ONB ✓

- protože $\hat{L}, \hat{K}$ jsou OSČBS je $\tilde{B}$ de facto bází v $\mathcal{H}$, a tedy $\forall f(\vec{x}) \in \mathcal{H}$:

$$f(\vec{x}) = \sum_{k=1}^{\infty} \langle f | \varphi_k \rangle \varphi_k(\vec{x}) \quad ✓$$

- označme: $\hat{L}\varphi_k = \lambda_k \varphi_k(\vec{x})$ & $\hat{K}\varphi_k = \mu_k \varphi_k(\vec{x})$
- nelze ale předpokládat, že $\lambda_k = \mu_k$ (to ani nemusí být pravda)
- pro složené operátory ale platí:

$$\hat{L}\hat{K}\varphi_k = \hat{L}\mu_k\varphi_k = \mu_k \hat{L}\varphi_k = \mu_k \lambda_k \varphi_k(\vec{x}) \quad \&$$

$$\hat{K}\hat{L}\varphi_k = \hat{K}\lambda_k\varphi_k = \lambda_k \hat{K}\varphi_k = \lambda_k \mu_k \varphi_k(\vec{x})$$

$\Rightarrow$ φ_k jsou též vlastní funkce obou složených operátorů

- chceme-li prokázat, že $\hat{K}$ & $\hat{L}$ komutují, je třeba $\forall f(\vec{x}) \in \mathcal{H}$ ukázat, že $\hat{L}\hat{K}f = \hat{K}\hat{L}f$
- výpočet tedy

$$\hat{L}\hat{K}f = \hat{L}\hat{K}\sum_{k=1}^{\infty} \langle f|\varphi_k\rangle \varphi_k(\vec{x}) \; ✓ = \left| \begin{array}{l} \hat{K} \text{ je omezený} \\ \Rightarrow \hat{K} \text{ je spojitý} \end{array} \right| =$$

$$= \hat{L}\left(\sum_{k=1}^{\infty} \langle f|\varphi_k\rangle \hat{K}\varphi_k\right) ✓ = \hat{L}\left(\sum_{k=1}^{\infty} \langle f|\varphi_k\rangle \mu_k \varphi_k(\vec{x})\right) ✓ = \left| \begin{array}{l} \hat{L} \text{ je omezený} \\ \Rightarrow \hat{L} \text{ je spojitý} \end{array} \right| =$$

$$= \sum_{k=1}^{\infty} \langle f|\varphi_k\rangle \hat{L}(\mu_k \varphi_k(\vec{x})) ✓ = \sum_{k=1}^{\infty} \langle f|\varphi_k\rangle \mu_k \lambda_k \; ✓$$

- analogicky: $\hat{K}\hat{L}f = \sum_{k=1}^{\infty} \langle f|\varphi_k\rangle \mu_k \lambda_k$ ✓

$\Rightarrow \forall f(\vec{x}) \in \mathcal{H}$: $\hat{L}\hat{K}f = \hat{K}\hat{L}f$ ✓ q.e.d.

$\mathcal{S}_+ = \{ \varphi(x) \in C^\infty(\mathbb{R}) : \operatorname{supp}(\varphi) \subset \mathbb{R}^+ \ \& \ \forall m,n \in \mathbb{N}_0 : \sup_{x\in\mathbb{R}} |x^m \varphi^{(n)}(x)| < +\infty \}$ ✓

- je treba ukázat, že $\forall \varphi(x) \in \mathcal{S}_+$ integrál $\int_{\mathbb{R}} x \cdot e^{-sx} \varphi(x)\,dx$ existuje

- na $\mathcal{S}_+$: $\displaystyle\int_{\mathbb{R}} x \cdot e^{-sx} \varphi(x)\,dx = \int_0^{\infty} x e^{-sx} \varphi(x)\,dx$

$$\forall x \in \mathbb{R}^+_0 : \quad |x e^{-sx} \varphi(x)| \overset{\text{pro } \operatorname{Re} s > 0}{\leq} |x \cdot \varphi(x)| \in \mathcal{L}(0; +\infty)$$ ✓

- pokud prokážeme, že $|x \cdot \varphi(x)| \in \mathcal{L}(0;+\infty)$, bude ze srovnávacího kritéria $x e^{-sx} \varphi(x) \in \mathcal{L}_1(\mathbb{R})$ ✓

- tedy to prokážeme:

a) $\displaystyle\int_0^{\alpha} |x \cdot \varphi(x)|\,dx = \int_{\langle 0;\alpha \rangle} |x \cdot \varphi(x)|\,dx$ existuje ✓

spojité funkce na kompaktu!

b) pro $x > \alpha > 0$: $|x^3 \cdot \varphi(x)| < K$ (z definice $\mathcal{S}_+$) ✓

$$|x \cdot \varphi(x)| < \underbrace{\frac{K}{x^2} \in \mathcal{L}(\alpha; +\infty)}_{\int_\alpha^{+\infty} \frac{K}{x^2}\,dx = \left[-\frac{K}{x}\right]_\alpha^{+\infty} = \frac{K}{\alpha}}$$ ✓

tj. $(\hat{T}\varphi)(s) := \int_{\mathbb{R}} x e^{-sx} \varphi(x)\,dx$ je dobře def pro $\varphi \in \mathcal{S}_+$ pro $\underline{\operatorname{Re} s > 0}$.

q.e.d.

II. varianta: ✓

a) $\operatorname{Dom}(\hat{T}) = \mathcal{P} \quad \Leftarrow \quad (f(x) \in \mathcal{P} \Rightarrow x \cdot f(x) \in \mathcal{P})$

b) $\operatorname{Dom}(\hat{T}) = \mathcal{D}_+ \quad \Leftarrow \quad (\mathcal{D} \subset \mathcal{S})$

c) $\operatorname{Dom}(\hat{T}) = \tilde{\mathcal{P}} \quad \Leftarrow \quad (f(x) \in \tilde{\mathcal{P}} \Rightarrow x \cdot f(x) \in \tilde{\mathcal{P}})$

d) ...

a) $\mathcal{T}$-obraz derivace   Re $s>0$

$$\mathcal{T}[\varphi'(x)] = \int_{\mathbb{R}} \varphi'(x)\, x \cdot e^{-sx}\, dx = \left| \begin{matrix} u = x\cdot e^{-sx} & v' = \varphi'(x) \\ u' = (1-sx)e^{-sx} & v = \varphi(x) \end{matrix} \right| =$$

$$= \left[x \cdot e^{-sx} \cdot \varphi(x)\right]_0^\infty + \int_0^\infty (sx-1)\, e^{-sx} \varphi(x)\, dx =$$

$$= 0 + s\int_0^\infty x e^{-sx} \varphi(x)\, dx - \int_0^\infty \varphi(x) e^{-sx}\, dx =$$

$$= s \cdot F(s) - \mathcal{L}[\varphi(x)]$$

Předpoklady:
- stačí $\varphi(x) \in \mathcal{S}_+$, protože $\mathcal{S}_+ \subset P$

b) mějme $s \in \mathbb{R}$, $s>0$

$$\frac{dF}{ds} = \frac{d}{ds}\int_0^{+\infty} x\, \varphi(x) e^{-sx}\, dx =$$

$|x^2 \cdot \varphi(x)|$ je integrabilní majoranta k derivovanému integrandu $\frac{\partial f}{\partial s}$, která je nezávislá na „s"

$\Leftarrow x^2\varphi(x) \in \mathcal{S}_+$ (zbytek prokázán v př. č. 1)

$$= -\int_0^\infty x^2 \varphi(x) e^{-sx}\, dx = -\mathcal{T}[x \cdot \varphi(x)]$$

$$\Rightarrow \frac{d^nF}{ds^n} = (-1)^n\, \mathcal{T}[x^n \cdot \varphi(x)]$$

Předpoklady:
$\varphi(x) \in \mathcal{S}_+$

c) to ulehčit

$$\int_0^\infty u \cdot f(u) \cdot G(u)\, du = \int_0^\infty u \cdot f(u) \int_0^\infty x \cdot g(x)\, e^{-ux}\, dx\, du =$$

$$= \int_0^\infty\int_0^\infty u \cdot x \cdot f(u)\, g(x)\, e^{-ux}\, dx\, du = \int_0^\infty x \cdot g(x) \int_0^\infty u \cdot f(u)\, e^{-ux}\, du\, dx =$$

$$= \int_0^\infty x \cdot F(x) \cdot g(x)\, dx$$

Předpoklady:
- pro aplikaci Fubiniovy věty je nezbytné, aby $x \cdot u \cdot f(u) \cdot g(x) \in \mathcal{L}(\mathbb{R}^+)$
- to lze ale zaručit tím, že $f(x), g(x) \in \mathcal{S}_+(\mathbb{R})$

$$12 \cdot \Theta(x) \cdot x^2 * \Theta(x) \cdot \cos^2(x) = f(x) * g(x)$$

$$\mathcal{L}[\Theta(x) \cdot x^2] = \frac{2}{p^3} \quad \wedge \quad \mathcal{L}[\Theta(x) \cos^2(x)] = \mathcal{L}\left[\Theta(x) \frac{1+\cos(2x)}{2}\right] =$$

$$= \frac{1}{2}\left(\frac{1}{p} + \frac{p}{p^2+4}\right)$$

$$\mathcal{L}[f(x) * g(x)] = \frac{24}{p^3} \cdot \frac{1}{2}\left(\frac{1}{p} + \frac{p}{p^2+4}\right) = 12\left(\frac{A}{p} + \frac{B}{p^2} + \frac{C}{p^3} + \frac{D}{p^4} + \frac{Ep+F}{p^2+4}\right)$$

$$(A, B, C, D, E, F) = \left(0, \frac{1}{4}, 0, 1, 0, -\frac{1}{4}\right)$$

$$12\,\Theta(x)\, x^2 * \Theta(x) \cos^2(x) = \mathcal{L}^{-1}\left[\frac{3}{p^2} + \frac{12}{p^4} - 3\,\frac{1}{p^2+4}\right] =$$

$$= \underline{3\,\Theta(x) \cdot x + 2\,\Theta(x) \cdot x^3 - \frac{3}{2}\Theta(x) \sin(2x)}$$

- není-li explicitně uveden výsledek obsahující Heavisideovu funkci ⇒ bod dolů

$$\lim_{\lambda\to\infty} \lambda^2 e^{-\lambda^2(x^2+y^2)} = ?$$

$$\lim_{\lambda\to\infty} \left(\lambda^2 e^{-\lambda^2(x^2+y^2)} ; \varphi(x,y)\right) = \lim_{\lambda\to\infty} \int_{\mathbb{R}^2} \lambda^2 e^{-\lambda^2(x^2+y^2)} \varphi(x,y)\, dx\, dy =$$

$$= \left| \begin{array}{l} \xi = \lambda x,\ \eta = y\lambda \\ \det \frac{D(\xi,\eta)}{D(x,y)} = \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} = \lambda^2 \end{array} \right\} \Rightarrow dx\,dy = \frac{1}{\lambda^2}\, d\xi\, d\eta \ \Big| =$$

$$= \lim_{\lambda\to\infty} \int_{\mathbb{R}^2} e^{-\xi^2-\eta^2} \varphi\left(\frac{\xi}{\lambda}, \frac{\eta}{\lambda}\right) d\xi\, d\eta = \left| \begin{array}{l} \text{nyní již existuje} \\ \text{integr. majoranta} \\ \text{nezávislá na parametru} \end{array} \right\} \text{je jí } e^{-(\xi^2+\eta^2)} \cdot K \cdot \Big| =$$

$$= \varphi(0,0) \cdot \int_{\mathbb{R}} e^{-\xi^2-\eta^2} d\xi\, d\eta = \varphi(0,0) \cdot \left(\int_{\mathbb{R}} e^{-\xi^2} d\xi \right)^2 = \varphi(0,0) \cdot \pi =$$

$$= \left(\pi\, \delta(x,y),\ \varphi(x,y) \right)$$

$$\mathcal{F}\left[\lambda^2 e^{-\lambda^2(x^2+y^2)}\right] = \lambda^2 \int_{\mathbb{R}} e^{-\lambda^2(x^2+y^2)} \cdot e^{i\xi x} \cdot e^{i\eta y}\, dx\, dy = \lambda^2 \sqrt{\frac{\pi}{\lambda^2}}\, e^{-\frac{\xi^2}{4\lambda^4}} \sqrt{\frac{\pi}{\lambda^2}}\, e^{-\frac{\eta^2}{4\lambda^4}} =$$

$$= \pi \cdot e^{-\frac{1}{4\lambda^4}(\xi^2+\eta^2)}$$

$$\lim_{\lambda\to\infty} \mathcal{F}\left[\lambda^2 e^{-\lambda^2(x^2+y^2)}\right] = \lim_{\lambda\to\infty} \pi \cdot e^{-\frac{1}{4\lambda^4}(\xi^2+\eta^2)} = \pi$$

$$\lim_{\lambda\to\infty} \left(e^{-\frac{1}{4\lambda^4}(\xi^2+\eta^2)},\ \varphi(\xi,\eta) \right) = \lim_{\lambda\to\infty} \int_{\mathbb{R}^2} e^{-\frac{1}{4\lambda^4}(\xi^2+\eta^2)} \varphi(\xi,\eta)\, d\xi\, d\eta =$$

$$= \left| \begin{array}{c} \text{integrabilní majoranta} \\ \varphi(\xi,\eta) \end{array} \right| = \int_{\mathbb{R}} \varphi(\xi,\eta)\, d\xi\, d\eta = \left(1,\ \varphi(\xi,\eta)\right)$$

$$q(x,y,z) = xy + xz + yz = \left| \begin{array}{l} x = \xi + \eta \\ y = \xi - \eta \\ z = \lambda \end{array} \right| = \xi^2 - \eta^2 + \xi\lambda + \eta\lambda + \xi\lambda - \eta\lambda =$$

$$= (\xi + \lambda)^2 - \lambda^2 - \eta^2 = a^2 - b^2 - c^2$$

$$\left.\begin{array}{l} a = \xi + \lambda \\ b = \lambda \\ c = \eta \end{array}\right\} \Rightarrow \left.\begin{array}{l} \xi = a - b \\ \eta = c \\ \lambda = b \end{array}\right\} \Rightarrow \begin{array}{l} x = a - b + c \\ y = a - b - c \\ z = b \end{array}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Polární báze: $B = \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}; \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right)$

Kontrola:

$$\left.\begin{array}{l} q(1,1,0) = 1 \\ q(-1,-1,1) = 1 - 1 - 1 = -1 \\ q(1,-1,0) = -1 \\ \det \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix} = -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2 \neq 0 \end{array}\right\} \boxed{\checkmark}$$

Transformační vztahy:

$$\begin{aligned} r &= x + y \\ s &= -x - y + z \\ t &= x - y \end{aligned}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} - \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \quad \& \quad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial r} - \frac{\partial f}{\partial s} - \frac{\partial f}{\partial t} \quad \& \quad \frac{\partial f}{\partial z} = \frac{\partial f}{\partial s}$$

Normální tvar:

$$\frac{\partial^2 f}{\partial r^2} - \frac{\partial^2 f}{\partial s^2} - \frac{\partial^2 f}{\partial t^2} + 2\frac{\partial f}{\partial t} = 0$$

– hyperbolická PDE

•) $\mathcal{S} \subset \mathcal{L}_1$ a pro funkce $f, g \in \mathcal{L}_1$ je dokázáno, že konvoluce existuje

• $\frac{\partial\varphi}{\partial x_l}$; $\frac{\partial\psi}{\partial x_k} \in \mathcal{S}(\mathbb{E}^n)$ ⇐ plyne přímo z definice $\mathcal{S}$ ✓✓

• Je ale $\varphi * \psi \in \mathcal{S}(\mathbb{E}^n)$?

a) je $\varphi * \psi$ nekonečně diferencovatelná?

$$\frac{\partial}{\partial x_k}(f*g) = \frac{\partial}{\partial x_k}\int_{\mathbb{R}^r} f(\vec{s})\cdot g(\vec{x}-\vec{s})\,d\vec{s} = \left|\ \text{půjde-li zaměnit}\ \right| =$$

$$= \int_{\mathbb{R}^r} f(\vec{s})\,\frac{\partial g(\vec{x}-\vec{s})}{\partial x_k}\,d\vec{s} \quad \left|\ \text{obě funkce z } \mathcal{S} \Rightarrow \Rightarrow \text{konvoluce existuje } \forall \vec{x}\ \right|$$

- postup lze opakovat pro další derivování
- ověřme ale možnost záměny
  - •) konvoluce existuje pro skoro všechna $\vec{x} \in \mathbb{E}^n$
  - •) $f(\vec{s})\, g(\vec{x}-\vec{s})$ je λ-měřitelná ⇐ obě funkce spojité
  - •) $\left| f(\vec{s})\,\frac{\partial g(\vec{x}-\vec{s})}{\partial x_k} \right| \le \underbrace{K \cdot |f(\vec{s})| \in \mathcal{L}(\mathbb{E}^n)}_{\Leftarrow\ f(\vec{s}) \in \mathcal{S}(\mathbb{E}^n)}$; $K \Leftarrow \frac{\partial g}{\partial x_k} \in \mathcal{S}$ ✓

b) je $\frac{\partial}{\partial x_k}(f*g)$ omezená

$$\left|\frac{\partial}{\partial x_k}(f*g)\right| = \left|\int_{\mathbb{E}^n} f(\vec{s})\,\frac{\partial g(\vec{x}-\vec{s})}{\partial x_k}\,d\vec{s}\right| \le K\cdot \overbrace{\int_{\mathbb{E}^n} |f(\vec{s})|\,d\vec{s}}^{\in \mathbb{R}}$$ ✓

c) je $x_k^m (f*g)$ omezená

$$|x_k^m (f*g)| = |x_k|^m \left|\int_{\mathbb{E}^n} f(\vec{s})\, g(\vec{x}-\vec{s})\,d\vec{s}\right| \le$$

$$\le \left|\ |x_k|^m \cdot |g(\vec{x}-\vec{s})| \le K\ \right| \le K\cdot \int_{\mathbb{E}^n} |f(\vec{s})|\,d\vec{s}$$ ✓

• Důkaz

$$\mathcal{F}\left[\frac{\partial^2}{\partial x_k \partial x_l}(\varphi*\psi)\right] = (-i\xi_k)(-i\xi_l)\,\mathcal{F}[\varphi]\cdot\mathcal{F}[\psi] = (-i\xi_l\,\mathcal{F}[\varphi])\cdot$$

$$\cdot(-i\xi_k\,\mathcal{F}[\psi]) = \mathcal{F}\left[\frac{\partial\varphi}{\partial x_l}\right]\cdot\mathcal{F}\left[\frac{\partial\psi}{\partial x_k}\right] = \mathcal{F}\left[\frac{\partial\varphi}{\partial x_l} * \frac{\partial\psi}{\partial x_k}\right]$$ ✓✓✓

$$\int_0^\infty \frac{e^{-a^2x}-e^{-b^2x}}{x^{3/2}}\,dx = ? \qquad \mathcal{L}[e^{-a^2x}\theta(x)] = \frac{1}{p+a^2} \quad \& \quad \mathcal{L}[e^{-b^2x}\theta(x)] = \frac{1}{p+b^2} \quad \checkmark$$

$$\mathcal{L}[\theta(x)\,x^\alpha] = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}} \quad \& \quad \alpha+1 \overset{!}{=} \frac{3}{2} \Rightarrow \alpha = \frac{1}{2} \quad \& \quad \Gamma\left(\tfrac{3}{2}\right) = \frac{\sqrt{\pi}}{2} \quad \checkmark$$

$$\mathcal{L}\left[\theta(x)\,\frac{\sqrt{x}}{\sqrt{\pi}}\cdot 2\right] = \frac{1}{p^{3/2}}$$

$$\int_0^\infty \frac{e^{-a^2x}-e^{-b^2x}}{x^{3/2}}\,dx = \int_0^\infty \frac{2}{\sqrt{\pi}}\cdot\sqrt{x}\left(\frac{1}{x+a^2}-\frac{1}{x+b^2}\right)dx \ \checkmark\checkmark = \left|\begin{array}{l} x = y^2 \\ dx = 2y\,dy \end{array}\right| =$$

$$= \frac{2}{\sqrt{\pi}}\cdot 2\int_0^\infty y^2\left(\frac{1}{y^2+a^2}-\frac{1}{y^2+b^2}\right)dy =$$

$$= \frac{4}{\sqrt{\pi}}\int_0^\infty \left(\frac{y^2+a^2-a^2}{y^2+a^2}-\frac{y^2+b^2-b^2}{y^2+b^2}\right)dy =$$

$$= \frac{4}{\sqrt{\pi}}\int_0^\infty \left(\frac{b^2}{y^2+b^2}-\frac{a^2}{y^2+a^2}\right)dy = \frac{4}{\sqrt{\pi}}\int_0^\infty \left(\frac{1}{1+\left(\frac{y}{b}\right)^2}-\frac{1}{1+\left(\frac{y}{a}\right)^2}\right)dy =$$

$$= \frac{4}{\sqrt{\pi}}\left[b\cdot\operatorname{arctg}\frac{y}{b} - a\cdot\operatorname{arctg}\frac{y}{a}\right]_0^\infty = \frac{4}{\sqrt{\pi}}\left(\frac{\pi}{2}b - \frac{\pi}{2}a\right) =$$

$$= 2\sqrt{\pi}\,(b-a) \qquad \checkmark\checkmark\checkmark\checkmark$$

Užijeme Sochockého vzorce: $$\frac{1}{x-i0} = \mathcal{P}\frac{1}{x} + i\pi\delta$$

Z linearity F-transf. pak: $$\mathcal{F}\left[\frac{1}{x-i0}\right] = \mathcal{F}\left[\mathcal{P}\frac{1}{x}\right] + \mathcal{F}[i\pi\delta] = i\pi + \mathcal{F}\left[\mathcal{P}\frac{1}{x}\right]$$

$$\left(\mathcal{F}\left[\mathcal{P}\tfrac{1}{x}\right], \varphi(\xi)\right) = \left(\mathcal{P}\tfrac{1}{x}; \mathcal{F}[\varphi(\xi)](x)\right) = \Big(\mathcal{P}\tfrac{1}{x}; \underbrace{\int_{\mathbb{R}} \varphi(\xi) e^{i\xi x} d\xi}_{=:\Omega(x)}\Big) =$$

$$= \int_0^\infty \frac{\Omega(x)-\Omega(-x)}{x} dx = \int_0^\infty \frac{\int_{\mathbb{R}} \varphi(\xi) e^{i\xi x} d\xi - \int_{\mathbb{R}} \varphi(\xi) e^{-i\xi x} d\xi}{x} dx = 2i \int_0^\infty \int_{\mathbb{R}} \frac{\sin(\xi x)}{x} \varphi(\xi)\, d\xi\, dx =$$

$$= \left| \begin{array}{l} \frac{\sin(\xi x)}{x}\varphi(\xi) \text{ spojitá s.v. v } \mathbb{R}^2 \\ \frac{\sin(\xi x)}{x} \in \mathcal{L}(0,+\infty) \end{array} \right| = 2i \int_{\mathbb{R}} \int_0^\infty \varphi(\xi) \frac{\sin(\xi x)}{x} dx\, d\xi =$$

$$= 2i \int_{\mathbb{R}} \varphi(\xi) \cdot \frac{\pi}{2} \cdot \operatorname{sgn}(\xi)\, d\xi = \left(i\pi \operatorname{sgn}(\xi); \varphi(\xi)\right)$$

$$\Rightarrow \quad \mathcal{F}\left[\frac{1}{x-i0}\right] = i\pi + i\pi \operatorname{sgn}(\xi) = i\pi(\operatorname{sgn}(\xi)+1) = 2\pi i\, \Theta(\xi)$$

•) Rovnost $\operatorname{sgn}(x)+1 = 2\Theta(x)$ není triviální, neboť v $\mathbb{R}$ tato rovnost neplatí ⇐

⇐ $1 = \operatorname{sgn}(0)+1 \neq 2\cdot\Theta(0) = 0$

$$(\operatorname{sgn}(x)+1, \varphi(x)) = \int_{\mathbb{R}} (\operatorname{sgn}(x)+1)\varphi(x)\, dx = \int_{\{0\}} \varphi(x)\, dx + \int_{(0;+\infty)} 2\varphi(x)\, dx =$$

$$= 0 + \int_0^{+\infty} 2\varphi(x)\, dx = \int_{\mathbb{R}} 2\Theta(x)\, dx$$

••) $$\int_0^\infty \frac{\sin(\beta x)}{x} dx = \left| \begin{array}{l} \text{pozor ale! není to banální úvaha,} \\ \text{protože } \frac{\sin(x)}{x} \in \mathcal{L}^* \setminus \mathcal{L}\ ! \end{array} \right| = \int_0^\infty \mathcal{L}^{-1}\left[\frac{1}{x}\right] \cdot \mathcal{L}[\sin(\beta x)]\, dp =$$

$$= \int_0^\infty \Theta(p) \cdot \frac{\beta}{p^2+\beta^2} dp = \int_0^\infty \frac{1}{\beta} \frac{1}{1+\left(\frac{p}{\beta}\right)^2} dp = \left[\operatorname{arctg}\frac{p}{\beta}\right]_0^\infty = \frac{\pi}{2}\cdot \operatorname{sgn}(\beta)$$

Dokažte, že je-li $g(\vec{x}) \in \mathcal{L}_{LOC}(\mathbb{E}^n)$, pak $\tilde{g}$ zavedený předpisem

$$(\tilde{g}; \varphi(\vec{x})) := \int_{\mathbb{E}^n} g(\vec{x})\varphi(\vec{x})\, d\vec{x}$$

je z $\mathcal{D}'(\mathbb{E}^n)$.

a) nejdůležitější je existence: ✓✗

- $supp(\varphi) \subset B_R \Rightarrow \int_{\mathbb{E}^n} g(\vec{x})\varphi(\vec{x})\,d\vec{x} = \int_{B_R} g(\vec{x})\varphi(\vec{x})\,d\vec{x}$

  B_R ... uzavřená koule o poloměru $R>0$, tedy kompakt

- $\varphi(\vec{x})$ je omezená na $\mathbb{E}^n \Rightarrow \exists K>0: \ |\varphi(\vec{x})| < K$ ✓

- $|g(\vec{x})\varphi(\vec{x})| \leq K \cdot |g(\vec{x})|$

  - víme ale, že $g(\vec{x}) \in \mathcal{L}(\Omega)$ pro každý kompakt Ω a z teorie Lebesgueova integrálu víme, že platí:

  $$\underbrace{g(\vec{x}) \in \Lambda_{\mu} \wedge g(\vec{x}) \in \mathcal{L}(A) \Rightarrow |g(\vec{x})| \in \mathcal{L}(A)}$$

  $$\Rightarrow K \cdot |g(\vec{x})| \in \mathcal{L}(B_R)$$ ✓✓✗

- ze srovnávacího kritéria pro Lebesgueův integrál pak:

  $$g(\vec{x})\varphi(\vec{x}) \in \mathcal{L}(B_R)$$

b) linearita funkcionálu je nyní již banální důsledek linearity Lebesgueova integrálu ✓

c) spojitost (uvažme $\varphi_k(\vec{x}) \rightrightarrows 0$)

$$\lim_{k\to\infty} (\tilde{g}; \varphi_k(\vec{x})) = \lim_{k\to\infty} \int_{\mathbb{E}^n} g(\vec{x})\varphi_k(\vec{x})\,d\vec{x} = \boxed{?} = \int_{\mathbb{E}^n} g(\vec{x}) \cdot \overbrace{\lim_{k\to\infty} \varphi_k(\vec{x})}^{=0}\,d\vec{x} = 0$$ ✓

Proč to ale lze?

$\varphi_k(x) \overset{E^n}{\rightrightarrows} 0$

kdyby $g(\vec{x})$ byla omezená na E^n, pak by také $g(\vec{x})\varphi_k(\vec{x}) \overset{E^n}{\rightrightarrows} 0$
a bylo by možné užít věty o záměně integrálu a limity
pro posloupnosti funkcí
- to ale obecně užít nelze

K záměně je nutno užít Lebesgueovy věty (o integrální majorantě)

a) $g(\vec{x})\varphi_k(\vec{x}) \in \mathcal{L}_n \Leftarrow g(\vec{x}) \in \mathcal{L}_{LOC}(E^n)$ &
& $\varphi_k(\vec{x}) \in C(E^n)$

b) z definice $\rightrightarrows$ existuje $R>0$ tak, že
$\operatorname{supp}(g\cdot\varphi_k) \subset B_R \quad (\forall k \in N)$

c) $$\int_{E^n} g(\vec{x})\varphi_k(\vec{x})\,d\vec{x} = \int_{B_R} g(\vec{x})\varphi_k(\vec{x})\,d\vec{x}$$

d) $\forall \vec{x} \in B_R$: $|g(\vec{x})\varphi_k(\vec{x})| \leq \underbrace{|g(\vec{x})|\cdot K}_{\Leftarrow\ B_R \text{ je kompakt } \&\ g(\vec{x}) \in \mathcal{L}_{LOC}} \in \mathcal{L}(B_R)$

existuje, neboť $\varphi_k \rightarrow 0$

- požadovanou integrabilní majorantou je tedy $K\cdot|g(\vec{x})|$

$$\hat{L} = \frac{\partial^2}{\partial t^2} + 2ai\frac{\partial^2}{\partial x \partial t} - a^2\frac{\partial^2}{\partial x^2} \quad \& \quad \hat{L}\mathcal{E}(x,t) = \delta(x,t) = \delta(x) \otimes \delta(t)$$ ✓

$$\frac{\partial^2 \mathcal{E}}{\partial t^2} + 2ai\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{E}}{\partial x}\right) - a^2\frac{\partial^2 \mathcal{E}}{\partial x^2} = \delta(x) \otimes \delta(t) \quad / \mathcal{F}_x$$

$$\frac{\partial^2 E}{\partial t^2} + 2ai\frac{\partial}{\partial t}\left(-i\xi E(\xi,t)\right) - a^2(-i\xi)^2 E(\xi,t) = 1(\xi) \otimes \delta(t)$$

$$\frac{\partial^2 E}{\partial t^2} + 2a\xi\frac{\partial E}{\partial t} + a^2\xi^2 E = \delta(t) \quad / \mathcal{L}_t$$ ✓

$$p^2\mathbb{E} + 2a\xi p\mathbb{E} + a^2\xi^2\mathbb{E} = 1$$

$$\mathbb{E}(\xi,p) = \frac{1}{p^2+2a\xi p+a^2\xi^2} = \frac{1}{(p+a\xi)^2}$$ ✓

$$E(\xi,t) = \mathcal{L}^{-1}\left[\frac{1}{(p+a\xi)^2}\right] = \theta(t)\cdot t\cdot e^{-a\xi t}$$ ✓✓

$$\mathcal{E}(x,t) = \mathcal{F}^{-1}\left[\theta(t)\cdot t\cdot e^{-a\xi t}\right] = \theta(t)\cdot t\cdot \mathcal{F}^{-1}\left[e^{-a\xi t}\right] =$$

$$= \theta(t)\cdot t\cdot \mathcal{F}^{-1}\mathcal{F}\mathcal{F}\left[\frac{1}{2\pi}e^{a\xi t}\right] = \frac{\theta(t)}{2\pi}\cdot t\cdot \mathcal{F}\left[e^{a\xi t}\right] =$$

$$= \left| ic = at \right| = \frac{\theta(t)}{2\pi}\cdot t\cdot 2\pi\delta(x+c) = \theta(t)\cdot t\cdot\delta(x-iat)$$

$$\underline{\mathcal{E}(x,t) = \theta(t)\cdot t\cdot\delta(x-i\cdot a t)}$$ ✓✓✓✓

$$\lim_{\lambda\to+\infty}\left(\widetilde{\theta(x)\lambda^2(1-\lambda x)e^{-\lambda x}}\,;\,\varphi(x)\right) = \left|\begin{array}{l}\text{zjevně se jedná o regulární distribuci,}\\ \text{neboť } g(x)=\theta(x)\lambda^2(1-\lambda x)e^{-\lambda x^2} \text{ je}\\ \text{spojité s.v. v } \mathbb{R}\end{array}\right| =$$

$$= \lim_{\lambda\to\infty}\int_0^\infty \lambda^2(1-\lambda x)e^{-\lambda x}\varphi(x)\,dx = \left|\begin{array}{ll} u=\varphi & v'=(1-\lambda x)e^{-\lambda x}\\ u'=\varphi' & v=xe^{-\lambda x}\end{array}\right| =$$

$$= \underbrace{\lim_{\lambda\to\infty}\left[\lambda^2\varphi(x)\cdot x\cdot e^{-\lambda x}\right]_0^\infty}_{=0\ (\text{why?})} - \lim_{\lambda\to\infty}\int_0^\infty \lambda^2\cdot x\cdot e^{-\lambda x}\cdot\varphi'(x)\,dx =$$

$$= -\lim_{\lambda\to\infty}\int_0^\infty \lambda^2\cdot x\cdot e^{-\lambda x}\varphi'(x)\,dx = \left|\begin{array}{l}\lambda x = y\\ dx=\frac{1}{\lambda}dy\end{array}\right| =$$

$$= -\lim_{\lambda\to\infty}\int_0^\infty \lambda^2\cdot\frac{y}{\lambda}\cdot e^{-y}\varphi'\!\left(\frac{y}{\lambda}\right)\frac{1}{\lambda}\,dy = -\lim_{\lambda\to\infty}\int_0^\infty y\cdot e^{-y}\cdot\varphi'\!\left(\frac{y}{\lambda}\right)dy =$$

$$= \left|\begin{array}{l}\text{diskuse možnosti}\\ \text{záměny!}\end{array}\right| = -\varphi'(0)\cdot\int_0^\infty y\,e^{-y}\,dy = -\varphi'(0) =$$

$$= (\tilde{\delta}'\,;\,\varphi(x))$$

$$\Rightarrow \lim_{\lambda\to\infty}\widetilde{\theta(x)\lambda^2(1-\lambda x)e^{-\lambda x}} = \tilde{\delta}'$$

$$y'' - 2y' + y = 3 - 4\int_0^x \sin(x-u)\cdot\cos(u)\,du \quad \& \quad y(0)=6 \quad \& \quad y'(0)=3$$

$$y'' - 2y' + y = 3 - 4\cdot \theta(x)\sin(x) * \theta(x)\cos(x) \quad \& \quad \mathcal{L}[y(x)] := Y(p)$$

$$\left.\begin{array}{l} \bullet\ \mathcal{L}[y'] = pY(p) - y(0) = p\cdot Y - 6 \\ \bullet\ \mathcal{L}[y''] = p(pY-6) - y'(0) = p^2Y - 6p - 3 \\ \bullet\ \mathcal{L}[3] = \frac{3}{p} \end{array}\right\} \checkmark\checkmark$$

$$\bullet\ \mathcal{L}[4\theta(x)\sin(x) * \theta(x)\cos(x)] = 4\mathcal{L}[\theta(x)\sin(x)]\cdot\mathcal{L}[\theta(x)\cos x] =$$

$$= \frac{4}{p^2+1}\cdot\frac{p}{p^2+1} = \frac{4p}{(p^2+1)^2} \quad \checkmark\checkmark$$

$$p^2Y - 6p - 3 - 2pY + 12 + Y = \frac{3}{p} + \frac{4p}{(p^2+1)^2}$$

$$(p-1)^2\cdot Y = -9 + \frac{3}{p} + 6p + \frac{4p}{(p^2+1)^2}$$

$$Y(p) = \frac{1}{(p-1)^2} + \frac{2}{p-1} + \frac{3}{p} - \frac{2}{(p^2+1)^2} + \frac{p}{p^2+1} \quad \checkmark\checkmark$$

$$y(x) = 3\theta(x) + 2\theta(x)e^x + \theta(x)\cdot x\cdot e^x + \theta(x)\cos(x) \underbrace{-2\mathcal{L}^{-1}\left[\frac{1}{(p^2+1)^2}\right]}_{\theta(x)[\sin x - x\cos x]}$$

$$y(x) = \theta(x)\left[3 + 2e^x + xe^x + \sin(x) + (1-x)\cos x\right] \quad \checkmark\checkmark$$

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$$\varphi(x) = \mu \int_0^x \frac{y^3}{x^3}\varphi(y)\,dy + 5x e^{\mu x}$$

$$K_1(x,y) = \frac{y^3}{x^3} \qquad K_{n+1}(x,y) = \int_y^x K_n(x,z)K(z,y)\,dz$$

$$K_2(x,y) = \int_y^x \frac{z^3}{x^3}\cdot\frac{y^3}{z^3}\,dz = \frac{y^3}{x^3}(x-y)$$

$$K_3(x,y) = \int_y^x \frac{z^3}{x^3}\cdot(x-z)\cdot\frac{y^3}{z^3}\,dz = \frac{y^3}{x^3}\left[-\frac{(x-z)^2}{2}\right]_y^x = \frac{1}{2}\frac{y^3}{x^3}(x-y)^2$$

$$K_4(x,y) = \frac{1}{2}\frac{y^3}{x^3}\int_y^x (x-z)^2\,dz = \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{y^3}{x^3}\left[-(x-z)^3\right]_y^x = \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{y^3}{x^3}(x-y)^3$$

$$K_l(x,y) = \frac{1}{(l-1)!}\frac{y^3}{x^3}(x-y)^{l-1}$$

indukce:

$$K_{l+1}(x,y) = \frac{1}{(l-1)!}\int_y^x \frac{z^3}{x^3}(x-z)^{l-1}\frac{y^3}{z^3}\,dz = \frac{1}{(l-1)!}\int_y^x (x-z)^{l-1}\,dz\,\frac{y^3}{x^3} =$$

$$= \frac{1}{(l-1)!}\frac{y^3}{x^3}\left[-\frac{(x-z)^l}{l}\right]_y^x = \frac{1}{l!}\frac{y^3}{x^3}(x-y)^l$$

precizní zápis!

$$R(x,y|\mu) = \sum_{l=0}^{\infty}\mu^l K_{l+1}(x,y) = \sum_{l=0}^{\infty}\mu^l\cdot\frac{1}{l!}\frac{y^3}{x^3}(x-y)^l =$$

$$= \frac{y^3}{x^3}e^{\mu(x-y)}$$

$$\varphi(x) = \mu\int_0^x R(x,y|\mu)f(y)\,dy + f(x) = \mu\int_0^x \frac{y^3}{x^3}e^{\mu(x-y)}5y e^{\mu y}\,dy + 5x e^{\mu x} =$$

$$= \frac{\mu}{x^3}e^{\mu x}\int_0^x 5y^4\,dy + 5x e^{\mu x} = \mu x^2 e^{\mu x} + 5x e^{\mu x} =$$

$$= x e^{\mu x}(\mu x + 5)$$

a) $\frac{1}{x+i0} = P\frac{1}{x} - i\pi\delta \Rightarrow \left(\frac{1}{x+i0}\right)' = \left(P\frac{1}{x}\right)' - i\pi\delta'$

$$\left(x^2 \left(P\frac{1}{x}\right)'; \varphi(x)\right) = \left(\left(P\frac{1}{x}\right)'; x^2\varphi(x)\right) = -\left(P\frac{1}{x}; 2x\varphi(x) + x^2\varphi'(x)\right) =$$

$$= -\left(P\frac{1}{x}; 2x\varphi(x)\right) - \left(P\frac{1}{x}; x^2\varphi'(x)\right) =$$

$$= -\, V_p\int_{\mathbb{R}} \frac{2x\varphi(x)}{x}\,dx - V_p\int_{\mathbb{R}} \frac{x^2\varphi'(x)}{x}\,dx =$$

$$= -2\int_{\mathbb{R}} \varphi(x)\,dx - \int_{\mathbb{R}} x\cdot\varphi'(x)\,dx = \left| \begin{matrix} u = x & v' = \varphi' \\ u' = 1 & v = \varphi \end{matrix} \right| =$$

$$= -2\int_{\mathbb{R}} \varphi(x)\,dx - \left[x\cdot\varphi(x)\right]_{-\infty}^{+\infty} + \int_{\mathbb{R}} \varphi(x)\,dx = -\int_{\mathbb{R}} \varphi(x)\,dx =$$

$$= (-1; \varphi(x))$$

$$(x\cdot\pi\cdot i\,\delta', \varphi(x)) = (\delta'; i\pi x\cdot\varphi(x)) = -(\delta; i\pi\varphi(x) + i\pi x\varphi'(x)) =$$

$$= -i\pi\varphi(0) = -(i\pi\delta; \varphi(x))$$

$\Rightarrow x^2\, i\pi\delta' = (-2i\pi)\, x\, \delta = \underline{0}$

$$\Rightarrow \quad x^2\cdot\left(\frac{1}{x+i0}\right)' = -1 + \cancel{i\pi\delta}$$

$(\tilde{h}; \varphi(x)) = \int_0^\infty \frac{\varphi(x)}{\sqrt{x}}\,dx \qquad x\cdot\tilde{h}' = ?$

a) existence $\int_0^\infty \frac{\varphi(x)}{\sqrt{x}}\,dx = \int_0^R \frac{\varphi(x)}{\sqrt{x}}\,dx = \underbrace{\int_0^\varepsilon \frac{\varphi(x)}{\sqrt{x}}\,dx}_{A} + \underbrace{\int_\varepsilon^R \frac{\varphi(x)}{\sqrt{x}}\,dx}_{B}$

b) linearita

c) spojitost

B … existuje vždycky (spojitá fce & kompakt)

A … $\left|\frac{\varphi(x)}{\sqrt{x}}\right| \leq \frac{K}{\sqrt{x}}$

je majoranta $K/\sqrt{x}$ integrabilní?

$\int_0^\varepsilon \frac{K}{\sqrt{x}}\,dx = \left[2K\sqrt{x}\right]_0^\varepsilon = 2K\cdot\varepsilon \in R$

„OK"

$\varphi_k(x) \rightrightarrows 0 \Rightarrow$ a) $D^\alpha\varphi_k(x) \overset{R}{\rightrightarrows} 0$

b) $\exists L>0: \quad \forall k\in N: \quad |\varphi_k(x)| < L$

$\lim_{k\to\infty}\int_0^\infty \frac{\varphi_k(x)}{\sqrt{x}}\,dx = \Big|$ hledáme majorantu (nezávislou na k), která je z $\mathcal{L}(0;+\infty)$ $\Big| =$

$= \left| \left|\frac{\varphi_k(x)}{\sqrt{x}}\right| \leq \frac{L}{\sqrt{x}} \right| = \int_0^\infty \lim_{k\to\infty} \frac{\varphi_k(x)}{\sqrt{x}}\,dx = \int_0^\infty 0\,dx = 0$ „OK"

$\Rightarrow \tilde{h} \in \mathcal{D}'(R)$

• navíc jsme také prokázali, že $\frac{\Theta(x)}{\sqrt{x}} \in \mathcal{L}_{loc}(R) \Rightarrow \tilde{h} = \frac{\Theta(x)}{\sqrt{x}} \in \mathcal{D}'_{reg}$

BONUS

• $(x\cdot h'; \varphi(x)) = (h', x\cdot\varphi(x)) = -(h; \varphi(x) + x\cdot\varphi'(x)) =$

$= -(h;\varphi(x)) - (h; x\cdot\varphi(x)) = -\int_0^\infty \frac{\varphi(x)}{\sqrt{x}}\,dx - \int_0^\infty \frac{x\cdot\varphi'(x)}{\sqrt{x}}\,dx =$

$= -\int_0^\infty \frac{\varphi(x)}{\sqrt{x}}\,dx - \int_0^\infty \sqrt{x}\,\varphi(x)\,dx = \begin{vmatrix} u=\sqrt{x} & v'=\varphi' \\ u' = \frac{1}{2}\frac{1}{\sqrt{x}} & v=\varphi \end{vmatrix} = -\int_0^\infty \frac{\varphi(x)}{\sqrt{x}}\,dx -$

$- \left[\sqrt{x}\,\varphi(x)\right]_0^\infty + \frac{1}{2}\int_0^\infty \frac{\varphi(x)}{\sqrt{x}}\,dx = -\frac{1}{2}\int_0^\infty \frac{\varphi(x)}{\sqrt{x}}\,dx = \left(-\frac{1}{2}h; \varphi(x)\right)$ $\boxed{x\cdot h' = -\frac{1}{2}h}$

$$\hat{L} = \frac{\partial^2}{\partial t^2} + 2ai\,\frac{\partial^2}{\partial x\partial t} - a^2\frac{\partial^2}{\partial x^2} \qquad \hat{L}\,\mathcal{E}(x,t) = \delta(x,t) = \delta(x)\otimes\delta(t)$$

$$\frac{\partial^2\mathcal{E}}{\partial t^2} + 2ai\,\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{E}}{\partial x}\right) - a^2\frac{\partial^2\mathcal{E}}{\partial x^2} = \delta(x)\otimes\delta(t) \qquad / \;\mathcal{F}_x$$

$$\frac{\partial^2 E}{\partial t^2} + 2ai\,\frac{\partial}{\partial t}\left(-i\xi E(\xi,t)\right) - a^2(-i\xi)^2E(\xi,t) = 1(\xi)\otimes\delta(t)$$

$$\frac{\partial^2 E}{\partial t^2} + 2a\xi\,\frac{\partial E}{\partial t} + a^2\xi^2E = \delta(t) \qquad / \;\mathcal{L}_t$$

$$p^2\mathbb{E} + 2a\xi p\,\mathbb{E} + a^2\xi^2\,\mathbb{E} = 1$$

$$\mathbb{E}(\xi,p) = \frac{1}{p^2+2a\xi p+a^2\xi^2} = \frac{1}{(p+a\xi)^2}$$

$$E(\xi,t) = \mathcal{L}^{-1}\left[\frac{1}{(p+a\xi)^2}\right] = \theta(t)\cdot t\cdot e^{-a\xi t}$$

$$\mathcal{E}(x,t) = \mathcal{F}^{-1}\left[\theta(t)\cdot t\cdot e^{-a\xi t}\right] = \theta(t)\cdot t\cdot\mathcal{F}^{-1}\left[e^{-a\xi t}\right] =$$

$$= \theta(t)\cdot t\cdot\mathcal{F}^{-1}\mathcal{F}\mathcal{F}\left[\frac{1}{2\pi}e^{a\xi t}\right] = \frac{\theta(t)}{2\pi}\cdot t\cdot\mathcal{F}\left[e^{a\xi t}\right] =$$

$$= \left| c = at \right| = \frac{\theta(t)}{2\pi}\cdot t\cdot 2\pi\,\delta(x+c) = \theta(t)\cdot t\cdot\delta(x-iat)$$

$$\mathcal{E}(x,t) = \theta(t)\cdot t\cdot\delta(x-i\cdot a\cdot t)$$

$$\int_0^\infty \frac{e^{-a^2x}-e^{-b^2x}}{x^{3/2}}\,dx = ? \qquad \mathcal{L}[e^{-a^2x}\theta(x)] = \frac{1}{p+a^2} \quad \& \quad \mathcal{L}[e^{-b^2x}\theta(x)] = \frac{1}{p+b^2}$$

$$\mathcal{L}[\theta(x)\,x^\alpha] = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}} \quad \& \quad \alpha+1 \stackrel{!}{=} \frac{3}{2} \Rightarrow \alpha = \frac{1}{2} \quad \& \quad \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\mathcal{L}\left[\theta(x)\,\frac{\sqrt{x}}{\sqrt{\pi}}\cdot 2\right] = \frac{1}{p^{3/2}}$$

$$\int_0^\infty \frac{e^{-a^2x}-e^{-b^2x}}{x^{3/2}}\,dx = \int_0^\infty \frac{2}{\sqrt{\pi}}\cdot\sqrt{x}\left(\frac{1}{x+a^2}-\frac{1}{x+b^2}\right)dx = \left|\begin{array}{l} x = y^2 \\ dx = 2y\,dy \end{array}\right| =$$

$$= \frac{2}{\sqrt{\pi}}\cdot 2\int_0^\infty y^2\left(\frac{1}{y^2+a^2}-\frac{1}{y^2+b^2}\right)dy =$$

$$= \frac{4}{\sqrt{\pi}}\int_0^\infty \left(\frac{y^2+a^2-a^2}{y^2+a^2}-\frac{y^2+b^2-b^2}{y^2+b^2}\right)dy =$$

$$= \frac{4}{\sqrt{\pi}}\int_0^\infty \left(\frac{b^2}{y^2+b^2}-\frac{a^2}{y^2+a^2}\right)dy = \frac{4}{\sqrt{\pi}}\int_0^\infty \left(\frac{1}{1+(\frac{y}{b})^2}-\frac{1}{1+(\frac{y}{a})^2}\right)dy =$$

$$= \frac{4}{\sqrt{\pi}}\left[b\cdot\operatorname{arctg}\frac{y}{b} - a\cdot\operatorname{arctg}\frac{y}{a}\right]_0^\infty = \frac{4}{\sqrt{\pi}}\left(\frac{\pi}{2}b-\frac{\pi}{2}a\right) =$$

$$= 2\sqrt{\pi}(b-a)$$

$$\hat{K} = \int_0^\infty e^{-y}(x+y)\,y \circ dy \qquad \hat{K}\varphi(x) = \lambda\varphi(x)$$

$$\int_0^\infty e^{-y}(x+y)\,y\cdot\varphi(y)\,dy = \lambda\varphi(x)$$

$$x\underbrace{\int_0^\infty e^{-y}y\,\varphi(y)\,dy}_{\alpha} + \underbrace{\int_0^\infty e^{-y}y^2\varphi(y)\,dy}_{\beta} = \lambda\varphi(x)$$

$$\lambda\varphi(x) = \alpha x + \beta$$

$$\lambda\cdot\alpha = \alpha\int_0^\infty x^2e^{-x}dx + \beta\int_0^\infty xe^{-x}dx \qquad \Bigg| \qquad \int_0^\infty x^ne^{-x}dx = n!$$

$$\lambda\cdot\beta = \alpha\int_0^\infty x^3e^{-x}dx + \beta\int_0^\infty x^2e^{-x}dx$$

$$\begin{aligned}\lambda\alpha &= 2\alpha+\beta\\ \lambda\beta &= 6\alpha+2\beta\end{aligned} \quad \Leftrightarrow \quad \begin{vmatrix} 2-\lambda & 1 \\ 6 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 6 \stackrel{!}{=} 0$$

$$\lambda_1 = 2+\sqrt{6} \quad \& \quad \lambda_2 = 2-\sqrt{6}$$

a) $\lambda_1 = 2+\sqrt{6}$ $\qquad (2+\sqrt{6})\alpha = 2\alpha+\beta \Rightarrow \beta = \sqrt{6}\,\alpha$

$$\varphi_1(x) = \Omega(x+\sqrt{6})$$

b) $\lambda_2 = 2-\sqrt{6}$ $\qquad (2-\sqrt{6})\alpha = 2\alpha+\beta \Rightarrow \beta = -\sqrt{6}\,\alpha$

$$\varphi_2(x) = \Phi(x-\sqrt{6})$$

$$\left(y^2 \frac{d^2\delta}{dy^2}; \varphi(y)\right) = \left(\frac{d^2\delta}{dy^2}; y^2\varphi(y)\right) = -\left(\frac{d\delta}{dy}; 2y\varphi(y) + y^2\frac{d\varphi}{dy}\right) =$$

$$= \left(\delta(y); 2\varphi(y) + 4y\frac{d\varphi}{dy} + y^2\frac{d^2\varphi}{dy^2}\right) = 2\varphi(0) = (2\delta(y); \varphi(y))$$

$$\lim_{\lambda\to\infty}\left(2\lambda x^3 e^{-\lambda^2x^2} P\frac{1}{x} \otimes \lambda^2\delta(y); \varphi(x,y)\right) = \lim_{\lambda\to\infty}\left(2\lambda x^3 e^{-\lambda^2x^2} P\frac{1}{x}; \left(\lambda^2\delta(y); \varphi(x,y)\right)\right) =$$

$$= \lim_{\lambda\to\infty}\left(2\lambda x^3 e^{-\lambda^2x^2} P\frac{1}{x}; \lambda^2\varphi(x,0)\right) = \lim_{\lambda\to\infty}\left(P\frac{1}{x}; 2\lambda x^3 e^{-\lambda^2x^2}\lambda^2\varphi(x,0)\right) =$$

$$= \lim_{\lambda\to\infty} V_p\int_R \frac{\lambda^3x^3e^{-\lambda^2x^2}\varphi(x,0)}{x}\,dx \;-2\cdot\lim_{\lambda\to\infty} V_p\int_R \lambda^3x^2e^{-\lambda^2x^2}\varphi(x,0)\,dx =$$

$$= 2\cdot\lim_{\lambda\to\infty}\int_R \lambda^3x^2e^{-\lambda^2x^2}\varphi(x,0)\,dx = \left|\begin{matrix} y=\lambda x \\ dy=\lambda dx\end{matrix}\right| = 2\cdot\lim_{\lambda\to\infty}\int_R y^2e^{-y^2}\varphi\left(\frac{y}{\lambda},0\right)dy =$$

$$= \left|\; \left|y^2e^{-y^2}\varphi\left(\frac{y}{\lambda},0\right)\right| \le K\cdot y^2e^{-y^2} \in \mathcal{L}(R)\;\right| = 2\cdot\int_R y^2e^{-y^2}\varphi(0,0)\,dy =$$

$$= 2\cdot\varphi(0,0)\int_R y^2e^{-y^2}dy = \left|\begin{matrix}\int_R e^{-ay^2}dy = \sqrt{\frac{\pi}{a}} \quad \Big/ \frac{d}{da} \\ -\int_R y^2e^{-ay^2}dy = -\frac{1}{2}\sqrt{\pi}\,a^{-3/2}\end{matrix}\right| = \varphi(0,0)\cdot\frac{\sqrt{\pi}}{2}\cdot 2 =$$

$$= \left(\sqrt{\pi}\,\delta(x,y); \varphi(x,y)\right)$$

$$\Rightarrow \lim_{\lambda\to\infty} g(x,y) = \sqrt{\pi}\,\delta(x,y) = \sqrt{\pi}\,\delta(x)\otimes \delta(y)$$

$$\theta_{\mu}(e^{\mu}-e^{x}) * \delta'' = ?$$

$$\left(\theta_{\mu}(e^{\mu}-e^{x}) * \delta''(x);\ \varphi(x)\right) \overset{\lim_{k\to\infty}}{=} \left(\theta_{\mu}(e^{\mu}-e^{x}) \otimes \delta''(y);\ \eta_k(x;y)\,\varphi(x+y)\right) =$$

$$= \lim_{k\to\infty} \left(\theta_{\mu}(e^{\mu}-e^{x});\ \left(\frac{d^2\delta}{dy^2};\ \eta_k(x;y)\,\varphi(x+y)\right)\right) =$$

$$= \lim_{k\to\infty} \left(\theta_{\mu}(e^{\mu}-e^{x});\ \left(\delta(y);\ \frac{\partial}{\partial y}\left[\frac{\partial \eta_k}{\partial y}\varphi(x+y) + \eta_k(x,y)\frac{\partial\varphi}{\partial y}(x+y)\right]\right)\right) =$$

$$= \lim_{k\to\infty} \left(\theta_{\mu}(e^{\mu}-e^{x});\ \left(\delta(y);\ \frac{\partial^2\eta_k}{\partial y^2}\varphi(x+y) + 2\frac{\partial\eta_k}{\partial y}\frac{\partial\varphi}{\partial y}(x+y) + \eta_k(x,y)\frac{\partial^2\varphi}{\partial y^2}(x+y)\right)\right) =$$

$$= \lim_{k\to\infty} \left(\theta_{\mu}(e^{\mu}-e^{x});\ \frac{\partial^2\eta_k}{\partial y^2}(x,0)\cdot\varphi(x) + 2\frac{\partial\eta_k}{\partial y}(x,0)\left.\frac{\partial\varphi}{\partial y}(x+y)\right|_{y=0} + \eta_k(x,0)\left.\frac{\partial^2\varphi}{\partial y^2}(x+y)\right|_{y=0}\right) =$$

$$= \lim_{k\to\infty} \left(\theta_{\mu}(e^{\mu}-e^{x});\ \frac{\partial^2\eta_k}{\partial y^2}(x,0)\varphi(x) + 2\frac{\partial\eta_k}{\partial y}(x,0)\cdot\frac{\partial\varphi}{\partial x} + \eta_k(x,0)\frac{\partial^2\varphi}{\partial x^2}\right) =$$

$$= \left(\theta_{\mu}(e^{\mu}-e^{x});\ \frac{\partial^2\varphi}{\partial x^2}\right) = -\left(\underbrace{\delta_{\mu}(e^{\mu}-e^{x})}_{=0} - \theta_{\mu}e^{x};\ \frac{\partial\varphi}{\partial x}\right) =$$

$$= \left(\theta_{\mu}e^{x};\ \frac{\partial\varphi}{\partial x}\right) = -\left(\delta_{\mu}e^{x} + \theta_{\mu}e^{x};\ \varphi(x)\right) = -\left(\delta_{\mu}e^{\mu} + \theta_{\mu}e^{x};\ \varphi(x)\right)$$

$$\Downarrow$$

$$\theta_{\mu}(e^{\mu}-e^{x}) * \delta'' = -e^{\mu}\cdot\delta_{\mu} + e^{x}\theta_{\mu}$$

$$\langle f|\varphi_n\rangle = \alpha_n \quad \wedge \quad \hat{L}\varphi_n = \lambda_n\varphi_n \quad \wedge \quad \mathrm{Re}^2(\lambda_n) + \mathrm{Im}^2(\lambda_n) = 4$$

$$\|f\|^2 = \langle f|f\rangle = \Big\langle \sum_{n=1}^{\infty}\alpha_n\varphi_n \Big| \sum_{m=1}^{\infty}\alpha_m\varphi_m \Big\rangle = \left| \begin{matrix}\text{spojitost}\\ \text{SS}\end{matrix} \right| = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_n\alpha_m^*\langle\varphi_n|\varphi_m\rangle =$$

$$= \sum_{n=1}^{\infty}|\alpha_n|^2\|\varphi_n\|^2 = \sum_{n=1}^{\infty}|\alpha_n|^2$$

$$\|\hat{L}f\| = \langle\hat{L}f|\hat{L}f\rangle = \Big\langle \hat{L}\Big(\sum_{n=1}^{\infty}\alpha_n\varphi_n\Big) \Big| \hat{L}\Big(\sum_{m=1}^{\infty}\alpha_m\varphi_m\Big)\Big\rangle = \left| \begin{matrix}\text{spojitost}\\ \text{operátoru}\end{matrix} \right| =$$

$$= \Big\langle \sum_{n=1}^{\infty}\hat{L}(\alpha_n\varphi_n) \Big| \sum_{m=1}^{\infty}\hat{L}(\alpha_m\varphi_m)\Big\rangle = \left| \begin{matrix}\text{spojitost}\\ \text{SS}\end{matrix} \right| =$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\langle\hat{L}(\alpha_n\varphi_n)|\hat{L}(\alpha_m\varphi_m)\rangle = \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\langle\alpha_n\hat{L}(\varphi_n)|\alpha_m\hat{L}(\varphi_m)\rangle =$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\langle\alpha_n\lambda_n\varphi_n|\alpha_m\lambda_m\varphi_m\rangle = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_n\lambda_n\alpha_m^*\lambda_m^*\langle\varphi_n|\varphi_m\rangle =$$

$$= \sum_{n=1}^{\infty}|\alpha_n|^2\cdot|\lambda_n|^2\cdot\|\varphi_n\|^2 = \sum_{k=0}^{\infty}|\alpha_{2k+1}|^2\cdot|\lambda_{2k+1}|^2 + \sum_{k=0}^{\infty}|\alpha_{2k}|^2|\lambda_{2k}|^2 =$$

$$= \sum_{k=0}^{\infty}4|\alpha_{2k+1}|^2 + \sum_{k=0}^{\infty}|\alpha_{2k}|^2 = \sum_{n=1}^{\infty}|\alpha_n|^2 + 3\sum_{k=0}^{\infty}|\alpha_{2k+1}|^2$$

$$\Rightarrow \qquad \frac{\|\hat{L}f\|}{\|f\|} = \frac{\sum_{n=1}^{\infty}|\alpha_n|^2 + 3\sum_{k=0}^{\infty}|\alpha_{2k+1}|^2}{\sum_{n=1}^{\infty}|\alpha_n|^2} = 1 + 3\,\frac{\sum_{k=0}^{\infty}|\alpha_{2k+1}|^2}{\sum_{n=1}^{\infty}|\alpha_n|^2} > 1$$

$\Rightarrow$ norma obrazu je větší než norma vzoru pro všechny nenulové funkce z $\mathcal{H}$

$$z'' + 3z' - 4\int_0^x z(s)\,ds = 8 \wedge z(0) = -2 \wedge z'(0) = 9.$$

```
In[43]:= Solve[p^2*Y+2p-9+3p*Y+6-4Y/p == 8/p, Y]
```

$$\text{Out[43]}= \left\{\left\{Y \to \frac{8+3p-2p^2}{(-1+p)\,(2+p)^2}\right\}\right\}$$

$$\text{In[44]}:= \text{InverseLaplaceTransform}\left[\frac{8+3p-2p^2}{(-1+p)\,(2+p)^2}, p, x\right] \text{// FullSimplify}$$

$$\text{Out[44]}= e^x + e^{-2x}\,(-3+2x)$$

$$\mathcal{L}[z(x)] = Z(p) \Rightarrow \mathcal{L}[z'] = pZ - z(0) = pZ + 2 \wedge$$

$$\wedge\ \mathcal{L}[z''] = p\,\mathcal{L}[z'] - z'(0) = p(pZ+2) - 9 \wedge$$

$$\wedge\ \mathcal{L}\left[\int_0^x z(s)\,ds\right] = \frac{Z}{p} \wedge \mathcal{L}[8] = \frac{8}{p}$$

$$p^2 Z + 2p - 9 + 3pZ + 6 - 4\frac{Z}{p} = \frac{8}{p}$$

$$Z = \frac{8+3p-2p^2}{(p-1)(p+2)^2} \Rightarrow Z(p) = \frac{1}{p-1} - \frac{3}{p+2} + \frac{2}{(p+2)^2}$$

$$z(x) = \theta(x)e^{x} - 3\,\theta(x)e^{-2x} + 2\,\theta(x)\,x\,e^{-2x} =$$

$$= \theta(x)\left[e^{x} - 3e^{-2x} + 2xe^{-2x}\right]$$

ZKOUŠKOVÁ PÍSEMKA Z PŘEDMĚTU RMF 4. 2. 2001

[1] Nalezněte funkci $\varphi(x)$, splňující rovnost

$$\varphi(x) = 2\int_0^\infty \left(x e^{-x^2-y^2} + x y^2 e^{-x^2}\right)\varphi(y)\,dy - \frac{1}{2} x e^{-x^2}$$

8 bodů

$$\varphi(x) = 2x e^{-x^2}\int_0^\infty e^{-y^2}\varphi(y)\,dy + 2x e^{-x^2}\int_0^\infty y^2\varphi(y)\,dy - \frac{1}{2} x e^{-x^2}$$

$$\varphi(x) = 2A x e^{-x^2} + 2B x e^{-x^2} - \frac{1}{2} x e^{-x^2}$$

- $A = 2A\int_0^\infty x e^{-2x^2}dx + 2B\int_0^\infty x e^{-2x^2}dx - \frac{1}{2}\int_0^\infty x e^{-2x^2}dx$
- $B = 2A\int_0^\infty x^3 e^{-2x^2}dx + 2B\int_0^\infty x^3 e^{-2x^2}dx - \frac{1}{2}\int_0^\infty x^3 e^{-2x^2}dx$

$$\int_0^\infty x e^{-2x^2}dx = \left|\begin{matrix} y = 2x^2 \\ dy = 4x\,dx\end{matrix}\right| = \int_0^\infty \frac{1}{4} e^{-y}dy = \frac{1}{4}\left[-e^{-y}\right]_0^\infty = \frac{1}{4}$$

$$\int_0^\infty x^3 e^{-2x^2}dx = \left|\begin{matrix} y = 2x^2 \\ dy = 4x\,dx\end{matrix}\right| = \int_0^\infty \frac{y}{2}\cdot\frac{1}{4} e^{-y}dy = \frac{1}{2}\int_0^\infty y e^{-y}dy = \left|\begin{matrix} u = y & v' = e^{-y} \\ u' = 1 & v = -e^{-y}\end{matrix}\right| =$$

$$= \frac{1}{8}\left[-y e^{-y}\right]_0^\infty + \frac{1}{2}\int_0^\infty e^{-y}dy = \frac{1}{2}$$

$$\left.\begin{matrix} A = 2A\,\tfrac{1}{4} + 2B\,\tfrac{1}{4} - \tfrac{1}{8} \\ B = 2A\,\tfrac{1}{2} + 2B\,\tfrac{1}{2} - \tfrac{1}{4}\end{matrix}\right\} \Rightarrow \begin{matrix} A = \frac{1}{2}A + \frac{1}{2}B - \frac{1}{8} \\ B = A + B - 1/4 \end{matrix}$$

$$A = 1/4 \quad \wedge \quad A = B - 1/4 \Rightarrow B = 2\cdot 1/4 = 1/2$$

$$(A, B) = (1/4,\ 1/2)$$

$$\varphi(x) = \frac{1}{2} x e^{-x^2} + x e^{-x^2} - \frac{1}{2} x e^{-x^2} = x e^{-x^2}$$

4

zkouška provedena!