6b

$y''' - y'' - 8y' + 12y = 100e^{2x}$

$\lambda^3 - \lambda^2 - 8\lambda + 12 = (\lambda+3)(\lambda-2)^2$ ✓

$F_S = \{e^{-3x}; e^{2x}; xe^{2x}\}$ ✓

$y_p(x) = \alpha x^2 e^{2x}$ ✓ $\cdot 12$

$y_p'(x) = \alpha(2x + 2x^2)e^{2x}$ $\cdot(-8)$

$y_p''(x) = \alpha(2 + 4x + 4x + 4x^2)e^{2x} = \alpha(2 + 8x + 4x^2)e^{2x}$ $\cdot(-1)$

$y_p'''(x) = \alpha(8 + 8x + 4 + 16x + 8x^2)e^{2x} = \alpha(12 + 24x + 8x^2)e^{2x}$ $\cdot 1$

x^0: $\alpha(-2 + 12) = 10\alpha \overset{!}{=} 100 \Rightarrow \alpha = 10$ ✓

x^1: $\alpha(-16 - 8 + 24) = 0$ ✓

x^2: $\alpha(12 - 16 + 4 + 8) = 0$ ✓

$y(x) = Ae^{-3x} + Be^{2x} + Cxe^{2x} + 10x^2e^{2x}$, $I = \mathbb{R}$ ✓

$$2x^2 - 2xy - 2xz + y^2 + 6yz - 2y + 13z^2 - 12z + 4 = 0$$

7b

$$q(x,y,z) = (y - x + 3z)^2 + x^2 + 4z^2 + 4xz =$$

$$= \underbrace{(y - x + 3z)^2}_{a} + \underbrace{(x + 2z)^2}_{b} + 0 \cdot \underbrace{z}_{c} \quad \checkmark$$

$$c = z$$

$$x = b - 2c$$

$$y = a + b - 2c - 3c = a + b - 5c$$

$$Q(a,b,c) = a^2 + b^2 - 2(a + b - 5c) - 12c + 4 =$$

$$= a^2 - 2a + b^2 - 2b - 2c + 4 =$$

$$= (a-1)^2 + (b-1)^2 - 2(c-1) \quad \checkmark$$

$$Q(\xi,\eta,\zeta) = \xi^2 + \eta^2 - 2\zeta$$

Název: eliptický paraboloid ✓

Normální tvar: $\xi^2 + \eta^2 - 2\zeta = 0$ ✓

Hlavní signatura: $(3,1,0)$

Vedlejší signatura: $(2,0,1)$ ✓

Kvadrika je regulární. ✓

9b

$y^2x^2 + yx^2(2x-3)y' = 5$ $\qquad y(1) = -3$

$x^2(2x-3)y' + x^2y = \frac{5}{y}$ $\qquad \cdot 2y$ (Bernoulli)

$y' + \frac{1}{2x-3}y = \frac{5}{x^2y(2x-3)}$ $\qquad \| \; v = y^2 \checkmark \Rightarrow v' = 2yy'$

$2yy' + \frac{2}{2x-3}y^2 = \frac{10}{x^2(2x-3)}$ ✓ $\|$ $\qquad \int \frac{2}{2x-3}dx = \ln|2x-3| + C$

$v' + \frac{2}{2x-3}v = \frac{10}{x^2(2x-3)}$ ✓ $\| \cdot (2x-3) \equiv$ I.F. ✓

$(v \cdot (2x-3))' = \frac{10}{x^2}$

$v \cdot (2x-3) = -\frac{10}{x} + \alpha$

$v(x) = \frac{\alpha x - 10}{x(2x-3)}$

$y(x) = \pm\sqrt{\frac{\alpha x - 10}{x(2x-3)}}$ ✓✓

$y(1) \overset{!}{=} -3 \qquad -3 = -\sqrt{\frac{\alpha - 10}{-1}}$

$10 - \alpha = 9$

$\alpha = 1$ ✓

$y(x) = \ominus\sqrt{\frac{10-x}{x(3-2x)}}$ ✓

$x > 0 \wedge 3 - 2x > 0 \wedge 10 - x \geq 0$
$x > 0 \wedge x < 3/2 \wedge x \leq 10$
$\Rightarrow I = Dom(y) = (0; \frac{3}{2})$ ✓

S ohledem na symetrii metriky (viz axiomy) musí být bod $\vec{a}$ středem svého okolí. Hledáme tedy střed kvadriky

5b

$$(x,y,z)\cdot\begin{pmatrix}9&2&-2\\2&1&0\\-2&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}-2\cdot(22,7,-3)\begin{pmatrix}x\\y\\z\end{pmatrix}=19$$

$A\cdot\vec{a}\overset{!}{=}\vec{b}$

$$\begin{pmatrix}9&2&-2\\2&1&0\\-2&0&1\end{pmatrix}\cdot\vec{a}=\begin{pmatrix}22\\7\\-3\end{pmatrix}\Rightarrow\quad\vec{a}=\begin{pmatrix}2\\3\\1\end{pmatrix}$$ ← střed

Hranice okolí je popsána rovnicí $\varrho(\vec{x},\vec{a})=\varepsilon$, tj.:

$$\|\vec{x}-\vec{a}\|^2=\varepsilon^2$$

$$\langle\vec{x}-\vec{a}\mid\vec{x}-\vec{a}\rangle=\varepsilon^2$$

$$\langle(x-2,y-3,z-1)\mid(x-2,y-3,z-1)\rangle=\varepsilon^2$$

$$(x-2,y-3,z-1)\cdot\begin{pmatrix}9&2&-2\\2&1&0\\-2&0&1\end{pmatrix}\begin{pmatrix}x-2\\y-3\\z-1\end{pmatrix}=\varepsilon^2$$

$$9x^2+4xy-4xz+y^2+z^2-44x-14y+6z+62=\varepsilon^2$$

$$\Rightarrow\quad\varepsilon^2-62\overset{!}{=}19$$

$$\varepsilon^2=81$$

$$\varepsilon=9$$

$$y'' + \frac{7}{x} y' + \frac{9}{x^2} y = \frac{24}{x^5 \cdot \ln^3(x)}$$

Eulerova ODR: $x = e^t \Rightarrow t = \ln(x)$

$$y' = \dot{y} \frac{1}{x} \quad \& \quad y'' = \ddot{y} \frac{1}{x^2} - \dot{y} \frac{1}{x^2}$$

Dosazení:

$$\ddot{y} - \dot{y} + 7\dot{y} + 9y = \frac{24}{e^{3t} \cdot t^3}$$

$$\ddot{y} + 6\dot{y} + 9y = \frac{24}{t^3} e^{-3t}$$

$$\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 \Rightarrow F_S = \{e^{-3t}; t e^{-3t}\}$$

$$W = \begin{vmatrix} e^{-3t} & t e^{-3t} \\ -3e^{-3t} & (1-3t)e^{-3t} \end{vmatrix} = e^{-6t}$$

$$\Delta_1 = \begin{vmatrix} 0 & t e^{-3t} \\ \frac{24}{t^3} e^{-3t} & \cdot \end{vmatrix} = -\frac{24}{t^2} e^{-6t}$$

$$\Delta_2 = \begin{vmatrix} e^{-3t} & 0 \\ \cdot & \frac{24}{t^3} e^{-3t} \end{vmatrix} = \frac{24}{t^3} e^{-6t}$$

$$\begin{cases} f_1(t) = -\frac{24}{t^2} & \& \quad f_2(t) = \frac{24}{t^3} \\ F_1(t) = 24\frac{1}{t} + \alpha & \& \quad F_2(t) = -\frac{12}{t^2} + \beta \end{cases}$$

$$y(t) = \alpha e^{-3t} + \beta t e^{-3t} + \frac{24}{t} e^{-3t} - \frac{12}{t^2} e^{-3t}$$

$$y(t) = \alpha e^{-3t} + \beta t e^{-3t} + \frac{12}{t} e^{-3t}$$

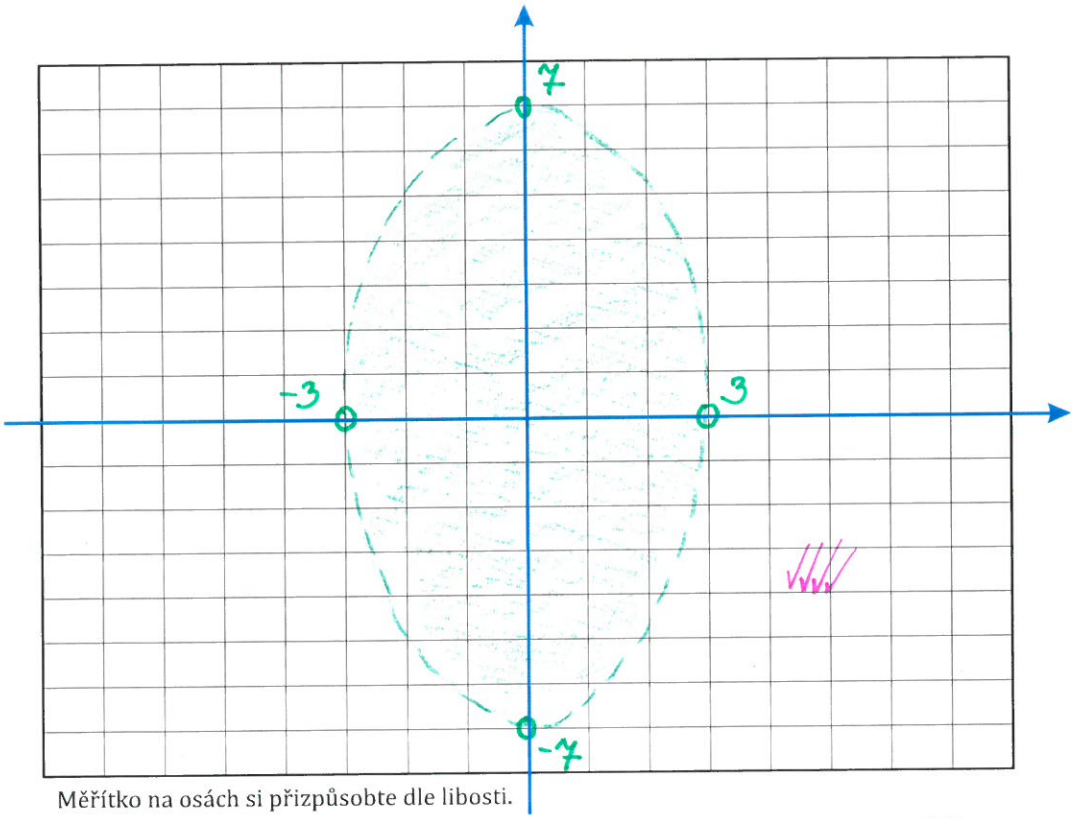
$$y(x) = \frac{\alpha}{x^3} + \frac{\beta}{x^3} \ln(x) + \frac{12}{x^3 \cdot \ln(x)}; \quad I = (1; +\infty)$$

4b

6 (bodů)

Do přiloženého obrázku vykreslete okolí bodu $(0, 0)$ o poloměru $\varepsilon = 7$. Metrika v Hilbertově prostoru $\mathbf{R}^2$ nechť je zadána skalárním součinem

$$(x_1, x_2)\begin{pmatrix} \frac{49}{9} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$



Měřítko na osách si přizpůsobte dle libosti.

$\langle \vec{x} | \vec{y} \rangle = \vec{x}^T A \vec{y}$ $\quad \|\vec{x} - \vec{0}\|^2 = \langle \vec{x} | \vec{x} \rangle = \vec{x}^T A \vec{x}$

$$\rho^2(\vec{x}, \vec{0}) = \|\vec{x} - \vec{0}\|^2 = (x_1, x_2)\begin{pmatrix} 49/9 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} < 49$$

$$\frac{49}{9} x_1^2 + x_2^2 < 49$$

$$\frac{x_1^2}{9} + \frac{x_2^2}{49} < 1$$

$$yx^2\left(y + y'(2x-7)\right) = -6$$

$$y'(2x-7) + y = -\frac{6}{x^2 y}$$

$$y' + \frac{1}{2x-7}y = -\frac{6}{x^2(2x-7)} \cdot \frac{1}{y} \quad /\!/ \text{ Bernoulli} \Rightarrow \text{nábožíme faktorem } 2y$$

$$2yy' + \frac{2}{2x-7}y^2 = -\frac{12}{x^2(2x-7)} \quad /\!/ \quad v = y^2 \Rightarrow v' = 2yy'$$

$$v' + \frac{2}{2x-7}v = -\frac{12}{x^2(2x-7)} \quad /\!/ \cdot (2x-7) = \text{I.F.}$$

$$\Uparrow \int \frac{2}{2x-7}\,dx = \ln|2x-7| + C$$

$$\frac{d}{dx}\left(v \cdot (2x-7)\right) = -\frac{12}{x^2}$$

$$v \cdot (2x-7) = \frac{12}{x} + \alpha$$

$$v(x) = \frac{\alpha x + 12}{x(2x-7)}$$

$$y(x) = \pm\sqrt{\frac{\alpha x + 12}{x(2x-7)}}$$

$$y(4) \overset{!}{=} -2 \qquad -2 = -\sqrt{\frac{4\alpha + 12}{4}}$$

$$\alpha + 3 = 4$$

$$\alpha = 1$$

$$y(x) = \ominus\sqrt{\frac{x+12}{x(2x-7)}}$$

$$\left.\begin{array}{l} x + 12 \geq 0 \wedge x > 0 \wedge 2x - 7 > 0 \\ x \geq -12 \wedge x > 0 \wedge x > 7/2 \end{array}\right\} \Rightarrow \underline{Dom(y) = I = \left(\tfrac{7}{2}; +\infty\right)}$$

$-8x + 5x^2 - 2y + 4xy + y^2 + 28z - 14xz - 2yz + 25z^2 = 4$ 7b

$$q(x,y,z) = (y + 2x - z)^2 + x^2 - 10xz + 24z^2 =$$
$$= \underbrace{(y + 2x - z)^2}_{a} + \underbrace{(x - 5z)^2}_{b} - \underbrace{z^2}_{c} \quad \checkmark$$

$$z = c$$
$$x = b + 5c$$
$$y = a + c - 2(b + 5c) = a - 2b - 9c$$

$$Q(x,y,z) = a^2 + b^2 - c^2 - 8(b + 5c) - 2(a - 2b - 9c) + 28c - 4 =$$
$$= a^2 - 2a + b^2 - 4b - c^2 + 6c - 4 =$$
$$= (a-1)^2 - 1 + (b-2)^2 - 4 - (c-3)^2 + 9 - 4 =$$
$$= (a-1)^2 + (b-2)^2 - (c-3)^2 = 0 \quad \checkmark$$

$$Q(\xi;\eta;\zeta) = \xi^2 + \eta^2 - \zeta^2 = 0$$

Název: kuželová plocha ✓
Signatury: $Sg(Q) = (2,1,1)$ & $sg(Q) = (2,1,0)$ ✓
Kvadrika je singulární. ✓
Normální tvar: $\xi^2 + \eta^2 - \zeta^2 = 0$ ✓

$$y''' - 6y'' + 32y = 72e^{4x}$$

$$(\lambda^3 - 6\lambda^2 + 32) : (\lambda + 2) = \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2$$ ✓

$$F_s = \{e^{-2x};\ e^{4x};\ xe^{4x}\}$$ ✓

$$y_p(x) = \alpha x^2 e^{4x}$$ ✓ $\cdot 32$

$$y_p'(x) = \alpha(2x + 4x^2)e^{4x} \quad \cdot 0$$

$$y_p''(x) = \alpha(2 + 8x + 8x + 16x^2)e^{4x} = \alpha(2 + 16x + 16x^2)e^{4x} \quad \cdot(-6)$$

$$y_p'''(x) = \alpha(16 + 32x + 8 + 64x + 64x^2)e^{2x} \quad \cdot 1$$

x^0: $\alpha(-12 + 24) = 12\alpha \stackrel{!}{=} 72 \quad \Rightarrow \quad \alpha = 6$ ✓

x^1: $\alpha(-96 + 96) = 0$ ✓

x^2: $\alpha(32 - 96 + 64) = 0$ ✓

$$y(x) = Ae^{-2x} + Be^{4x} + Cxe^{4x} + 6x^2e^{4x};\ I = \mathbb{R}$$

5b

$\vec{a}$ je středem svého okolí $U_\varepsilon(\vec{a}) = E$

$$A \cdot \vec{a} = \vec{b} \qquad \begin{pmatrix} 4 & 2 & -2 \\ 2 & 2 & 0 \\ -2 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \overset{!}{=} \begin{pmatrix} -6 \\ -8 \\ -4 \end{pmatrix}$$

$$\vec{a} = \begin{pmatrix} -1 \\ -3 \\ -2 \end{pmatrix} \leftarrow \text{střed}$$

Hranice okolí: $\varrho(\vec{x}, \vec{a}) = \varepsilon$, tj.

$$\|\vec{x} - \vec{a}\|^2 = \varepsilon^2$$

$$\langle \vec{x} - \vec{a} \mid \vec{x} - \vec{a} \rangle = \varepsilon^2$$

$$(x+1, y+3, z+2) \cdot \begin{pmatrix} 4 & 2 & -2 \\ 2 & 2 & 0 \\ -2 & 0 & 3 \end{pmatrix} \begin{pmatrix} x+1 \\ y+3 \\ z+2 \end{pmatrix} = \varepsilon^2$$

$$4x^2 + 2y^2 + 3z^2 + 4xy - 4xz + 12x + 16y + 8z + 38 = \varepsilon^2$$

$$\Rightarrow \quad \varepsilon^2 - 38 = 83$$

$$\varepsilon^2 = 121$$

$$\varepsilon = 11$$

$$y'' + 5\frac{y'}{x} + 4\frac{y}{x^2} = \frac{48}{x^4 \cdot \ln^3(x)} \quad \Rightarrow \quad x \overset{?}{>} 1 \quad \Rightarrow I = (1; +\infty)$$

Eulerova ODR: $x = e^t \Rightarrow t = \ln(x)$

$y' = \dot{y}\frac{1}{x}$ & $y'' = \ddot{y}\frac{1}{x^2} - \dot{y}\frac{1}{x^2}$ ✓

9b

Dosazení:

$$\ddot{y} - \dot{y} + 5\dot{y} + 4y = \frac{48}{e^{2t} \cdot t^3}$$

$$\ddot{y} + 4\dot{y} + 4y = \frac{48}{t^3}e^{-2t}$$ ✓

$$\lambda^2 + 4\lambda + 4 = (\lambda+2)^2 \Rightarrow F_1 = \{e^{-2t}; te^{-2t}\}$$ ✓

$$W = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} = e^{-4t}$$

$$A_1 = \begin{vmatrix} 0 & te^{-2t} \\ \frac{48}{t^3}e^{-2t} & \cdot \end{vmatrix} = -\frac{48}{t^2}e^{-4t}$$

$$A_2 = \begin{vmatrix} e^{-2t} & 0 \\ \cdot & \frac{48}{t^3}e^{-2t} \end{vmatrix} = \frac{48}{t^3}e^{-2t}$$ ✓

$$\begin{cases} f_1(t) = -\frac{48}{t^2} & \& \quad f_2(t) = \frac{48}{t^3} \\ F_1(t) = \frac{48}{t} + \alpha & \& \quad F_2(t) = -\frac{24}{t^2} + \beta \end{cases}$$ ✓

$$y(t) = \alpha e^{-2t} + \beta \cdot t \cdot e^{-2t} + \frac{48}{t}e^{-2t} - \frac{24}{t^2}te^{-2t}$$

$$y(t) = \alpha e^{-2t} + \beta \cdot t \cdot e^{-2t} + \frac{24}{t}e^{-2t}$$ ✓

$$y(x) = \frac{\alpha}{x^2} + \frac{\beta}{x^2}\ln(x) + \frac{24}{x^2 \cdot \ln(x)}$$ ✓

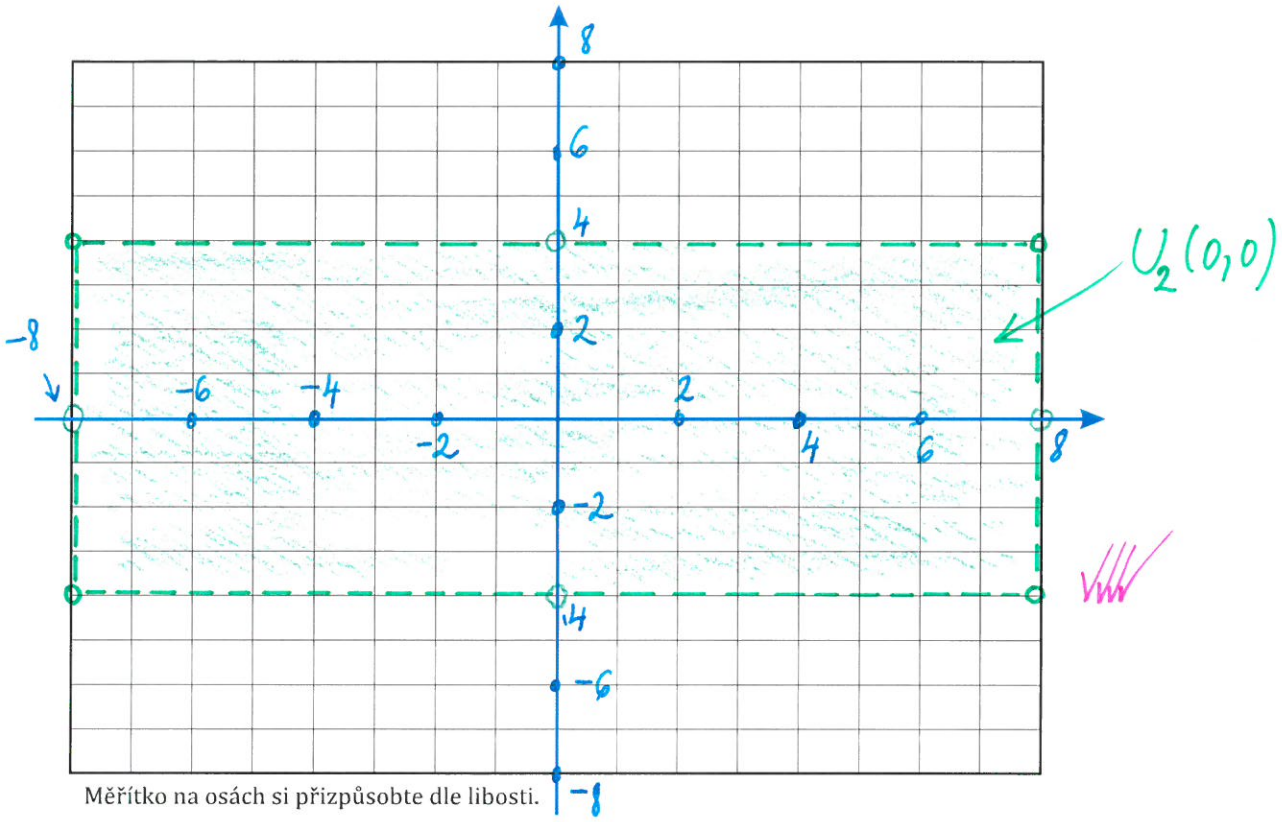
$$I = (1; +\infty)$$ ✓

4b

6 (bodů)

Do přiloženého obrázku vykreslete okolí bodu $(0,0)$ o poloměru $\varepsilon = 2$. Metrika v $\mathbf{R}^2$ je zadána předpisem

$$\varrho(\vec{x},\vec{y}) = \max\left\{\frac{1}{4}|x_1 - y_1|, \frac{1}{2}|x_2 - y_2|\right\}.$$



Měřítko na osách si přizpůsobte dle libosti.

$\max\left\{\frac{1}{4}|x_1 - 0|; \frac{1}{2}|x_2 - 0|\right\} < 2$

$\max\left\{\frac{1}{8}|x_1|; \frac{1}{4}|x_2|\right\} < 1$