1.

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- $\mathrm{Dom}(g) = \mathbb{R}$ & $\mathrm{supp}(g) = (0; +\infty)$ ... OBA OK
- $\mathrm{Ran}(g) \subset \langle 0; +\infty)$ ... funkce je zjevně nezáporná
- Integrabilita ... $g(x) \overset{?}{\in} \mathcal{L}(\mathbb{R})$
  - nejprve ukážeme, že $g(x) \in C((-\infty; x_0\rangle)$ ... $\forall x_0 \in \mathbb{R}$
  - jediný problém je v nule
  - jinak triviální
  - v té nule:

$$\lim_{x\to 0_+} \frac{1}{\sqrt{x^3}} e^{-\frac{\lambda x}{2\mu^2}} \cdot e^{\frac{\lambda}{\mu}} \cdot e^{-\frac{\lambda}{2x}} =$$

$$= \lim_{x\to 0_+} \frac{1}{\sqrt{x^3}} e^{-\frac{\lambda}{2x}} \cdot e^{\lambda/\mu} = \left\| y = \frac{1}{x} \right\| =$$

$$= e^{\lambda/\mu} \lim_{y\to+\infty} y^{3/2} \cdot e^{-\frac{\lambda}{2}\cdot y} = 0$$

& $\lim_{x\to 0_-} g(x) = 0$ triviálně

  - spojitá funkce (s pozitivním nosičem) je integrabilní na každém $(-\infty; x_0\rangle$
  - na $\langle x_0; +\infty)$ platí:

$$\frac{1}{\sqrt{x^3}} e^{-\frac{\lambda x}{2\mu^2}} \cdot e^{\lambda/\mu} e^{-\frac{\lambda}{2x}} \leqq \underbrace{1 \cdot e^{-\frac{\lambda x}{2\mu^2}} \cdot e^{\frac{\lambda}{\mu}} \cdot 1}_{\text{ta je ale integrabilní}}$$

  - ze srovnávacího kritéria: $g(x) \in \mathcal{L}(\mathbb{R})$

- balancovanost

$$\lim_{x\to+\infty} \frac{\ln(g(x))}{x} = \lim_{x\to\infty} \frac{\ln x^{-3/2}}{x} - \lim_{x\to\infty} \frac{\lambda x}{2\mu^2}\cdot\frac{1}{x} + \lim_{x\to+\infty} \frac{\lambda}{\mu x} -$$

$$- \lim_{x\to+\infty} \frac{\lambda}{2x^2} = -\frac{3}{2}\lim_{x\to\infty}\frac{\ln x}{x} - \frac{\lambda}{2\mu^2} + 0 - 0 \overset{L'H}{=} -\frac{3}{2}\lim_{x\to+\infty}\frac{1}{x} - \frac{\lambda}{2\mu^2} =$$

$$= -\frac{\lambda}{2\mu^2} < 0 \Rightarrow g(x) \in \mathcal{B}\left[\frac{\lambda}{2\mu^2}\right] \Rightarrow \mathrm{int}(g) = \frac{\lambda}{2\mu^2}$$

2. $\mathcal{L}[g(x)] = \int_{\mathbb{R}} g(x)\, e^{-sx}\, dx \triangleq G(s)$

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$$s \cdot \mathcal{L}[h(x)] = s \cdot \int_{\mathbb{R}} h(x)\, e^{-sx}\, dx = s \cdot \int_{\mathbb{R}} \theta(x) \int_{x}^{+\infty} g(y)\, dy\, e^{-sx}\, dx =$$

$$= \int_{\mathbb{R}^2} \theta(x) \cdot s \cdot \theta(y-x) \cdot g(y)\, e^{-sx}\, d(x,y) =$$

$$= \int_{\mathbb{R}} g(y) \int_{\mathbb{R}} \theta(x)\, \theta(y-x) \cdot e^{-sx} \cdot s\, \overbrace{dx\, dy}^{\text{FUBINI}} = \left\| \begin{matrix} y - x > 0 \\ x < y \end{matrix} \right\| =$$

$$= \int_{\mathbb{R}} g(y) \int_{0}^{y} s \cdot e^{-sx}\, dx\, dy =$$

$$= \int_{\mathbb{R}} g(y) \left[-e^{-sx} \right]_0^y dy = \int_{\mathbb{R}} g(y) \cdot \left[1 - e^{-sy} \right] dy =$$

$$= \int_{\mathbb{R}} g(y)\, dy - \int_{\mathbb{R}} g(y)\, e^{-sy}\, dy = \|g\| - G(s)$$

$$\Rightarrow \quad H(s) = \frac{\|g\| - G(s)}{s} \quad (s > 0)$$

3. Užijeme: $\int_{R} g(x)\cdot H(x)\,dx = \int_{R} G(x)\cdot h(x)\,dx$

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~~(4 body)~~

•) $H(s) = \frac{1}{s^3}$ & $h(x) = \mathcal{L}^{-1}[1/s^3] = ?$

$\mathcal{L}[\theta(x)] = \frac{1}{s} \quad \Big| \frac{d^2}{ds^2}$

$\mathcal{L}[\theta(x)\cdot x^2] = \frac{2}{s^3} \quad \Rightarrow \quad h(x) = \frac{1}{2}\theta(x)\cdot x^2$

✓

•) $g(x) = \sin(x) + x\cdot\cos x$

$\mathcal{L}[\sin(x)] = \frac{1}{s^2+1}$

$\mathcal{L}[\cos(x)] = \frac{s}{s^2+1} \quad \Rightarrow \quad \mathcal{L}[x\cdot\cos(x)] = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{s^2+1-2s^2}{(s^2+1)^2}(-1)$

$= \frac{s^2-1}{(s^2+1)^2}$

$\mathcal{L}[\sin(x) - x\cdot\cos(x)] = \frac{2}{(s^2+1)^2}$

✓✓

Výpočet:

$\int_0^{+\infty} \frac{\sin(x) - x\cos(x)}{x^3}\,dx = \int_0^{+\infty} \frac{x^2}{(1+x^2)^2}\,dx$ ✓ $= \left\| \begin{matrix} u = x & v' = \frac{x}{(1+x^2)^2} \\ u' = 1 & v = -\frac{1}{2}\frac{1}{(1+x^2)} \end{matrix} \right\| =$ ✓

$= \left[-\frac{1}{2}\frac{x}{1+x^2}\right]_0^{+\infty} + \frac{1}{2}\int_0^{+\infty}\frac{1}{1+x^2}\,dx =$

$= \frac{1}{2}\left[\operatorname{arctg} x\right]_0^{+\infty} = \pi/4$ ✓

4. $f(x) = \Theta(x)\,\frac{\lambda^{n+1}}{n!}\,x^n e^{-\lambda x}$ 6b

$$F(s) = \mathcal{L}[f(x)] = \frac{\lambda^{n+1}}{n!}\mathcal{L}[x^n e^{-\lambda x}] = \frac{\lambda^{n+1}}{n!}(-1)^n \frac{d^n}{ds^n}\mathcal{L}[e^{-\lambda x}] =$$

$$= \frac{\lambda^{n+1}}{n!}(-1)^n \frac{d^n}{ds^n}\left[\mathcal{L}[\Theta(x)]\Big|_{s\to s+\lambda}\right] =$$

$$= \frac{\lambda^{n+1}}{n!}(-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s+\lambda}\right) = \frac{\lambda^{n+1}}{n!}\,\frac{n!}{(s+\lambda)^{n+1}} = \left(\frac{\lambda}{\lambda+s}\right)^{n+1}$$

$$G(s) = \left(\frac{\lambda}{\lambda+s}\right)^{m+1}$$

•) výpočet:

$$\mathcal{L}[g(x) * f(x)] = F(s)\cdot G(s) = \left(\frac{\lambda}{\lambda+s}\right)^{m+n+2}$$

$$g(x) * f(x) = \mathcal{L}^{-1}\left[\left(\frac{\lambda}{\lambda+s}\right)^{m+n+2}\right] = \Theta(x)\,\frac{\lambda^{m+n+2}}{(m+n+1)!}\,x^{m+n+1}e^{-\lambda x}$$