6

In[4]:= Integrate[(x - a * x^2)^2 * x^2, {x, -1, 1}]

Out[4]= $\frac{2}{5} + \frac{2a^2}{7}$

In[28]:= Solve[Sqrt[$\frac{2}{5} + \frac{2a^2}{7}$] == 3 / Sqrt[10], a]

Out[28]= $\left\{\left\{a \to -\frac{\sqrt{7}}{2}\right\}, \left\{a \to \frac{\sqrt{7}}{2}\right\}\right\}$

váha skalárního součinu

$$\rho^2(h,m) = \|h-m\|^2 = \langle h-m \mid h-m \rangle = \int_{-1}^{1} (h-m)^2 \cdot x^2 \, dx =$$

$$= \int_{-1}^{1} (x-ax^2)^2 x^2 \, dx = \int_{-1}^{1} (x^2 - 2ax^3 + a^2x^4)\, x^2 \, dx =$$

$$= \int_{-1}^{1} (x^4 + a^2 x^6)\, dx = \left[\frac{x^5}{5} + a^2 \frac{x^7}{7}\right]_{-1}^{1} = 2\cdot\left(\frac{1}{5} + a^2 \frac{1}{7}\right)$$

$$\rho^2(h,m) \overset{!}{=} \left(\frac{3}{\sqrt{10}}\right)^2 = \frac{9}{10}$$

$$\frac{2}{5} + \frac{2}{7}a^2 = \frac{9}{10}$$

$$\frac{2}{7}a^2 = \frac{5}{10}$$

$$a^2 = \frac{7}{4}$$

$$a = \pm\frac{\sqrt{7}}{2}$$

$$2y' = \frac{y}{x} - 3 - \frac{9x}{3x+y} \qquad y\left(\tfrac{2}{3}\right) = -2$$

- jde o homogenní rovnici $\Rightarrow$ $\underline{y(x) = x \cdot z(x)}$ $\Rightarrow$ $y' = z + xz'$

$$2z + 2xz' = z - 3 - \frac{9x}{3x+xz} = z - 3 - \frac{9}{3+z}$$

$$-2xz' = z + 3 + \frac{9}{3+z}$$

$$-2xz' = \frac{z^2+6z+18}{3+z}$$

$$\int \frac{2z+6}{z^2+6z+18}\,dz = -\int \frac{1}{x}\,dx$$

$$\ln|z^2+6z+18| = -\ln|x| + C$$

$$z^2+6z+18 = \frac{C}{x}$$

$$\underline{\underline{\frac{y^2}{x^2} + 6\frac{y}{x} + 18 = \frac{C}{x}}}$$

$$y^2 + 6xy + 18x^2 + Dx = 0 \qquad D = ?$$

$$4 + 6\cdot\tfrac{2}{3}\cdot(-2) + 18\cdot\tfrac{4}{9} + D\tfrac{2}{3} = 0$$

$$4 - 8 + 8 + \frac{2}{3}D \overset{!}{=} 0 \quad \Rightarrow \quad D = -6$$

$$\underline{\underline{y^2 + 6xy + 18x^2 - 6x = 0}}$$

$$(y+3x)^2 + 9x^2 - 6x = 0$$

$\Rightarrow$ jde skutečně o elipsu

$$A = \begin{pmatrix} 18 & 3 \\ 3 & 1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 18 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \underline{\underline{(s_1, s_2) = \left(\tfrac{1}{3}, -1\right)}}$$

(konkrétní řešení nebudu hledat)

2. $\alpha \stackrel{!}{=} \beta$ ⇐ kvůli symetrii! 6b

$q(\vec{x}, \vec{y})$ musí být pozitivně definitní ⇒ všechna vlastní čísla matice A musejí být kladná ⇐ slovní formulace (nutná!)

$$\begin{vmatrix} 4-\lambda & 0 & 0 & \alpha \\ 0 & 2-\lambda & 0 & 0 \\ 0 & 0 & 3-\lambda & 0 \\ \beta & 0 & 0 & 4-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 4-\lambda & 0 & \beta \\ 0 & 3-\lambda & 0 \\ \beta & 0 & 4-\lambda \end{vmatrix} =$$

$$= (2-\lambda)(3-\lambda)\begin{vmatrix} 4-\lambda & \beta \\ \beta & 4-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)\left((4-\lambda)^2 - \beta^2\right) =$$

$$= (2-\lambda)(3-\lambda)(16 - 8\lambda + \lambda^2 - \beta^2)$$

Řešení: $\lambda^2 - 8\lambda + 16 - \beta^2 = 0$

$$\lambda_{1,2} = \frac{8 \pm \sqrt{64 - 4(16-\beta^2)}}{2} = 4 \pm |\beta|$$

– požadujeme, aby $\underbrace{4 + |\beta| > 0 \wedge 4 - |\beta| > 0}_{|\beta| < 4}$

$\beta \in (-4, 4)$ & $\alpha = \beta$

– lze také Sylvesterovým kritériem $\Delta_1 = 4 > 0$ $\Delta_2 = \begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} = 8 > 0$

$\Delta_3 = \begin{vmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 24 > 0$

snazší cesta

$$\det A = 2\begin{vmatrix} 4 & 0 & \beta \\ 0 & 3 & 0 \\ \beta & 0 & 4 \end{vmatrix} = 6 \cdot \begin{vmatrix} 4 & \beta \\ \beta & 4 \end{vmatrix} = 6(16 - \beta^2) \stackrel{!}{>} 0$$

$$16 - \beta^2 > 0 \Rightarrow \beta \in (-4, 4)$$

! numerické chyby se netolerují !

9b.

$$x^2 - 2xy + 2y^2 + 8xz - 16yz + 32z^2 = (x-y+4z)^2 + y^2 + 16z^2 - 8yz =$$
$$= (x-y+4z)^2 + (y-4z)^2$$

$$\left.\begin{aligned} \xi &= x-y+4z \\ \eta &= y-4z \\ \lambda &= z \end{aligned}\right\} \Rightarrow \begin{aligned} x &= \xi+\eta+4\lambda-4\lambda = \xi+\eta \\ y &= \eta+4\lambda \\ z &= \lambda \end{aligned}$$

$$Q(x,y,z) = \xi^2 + \eta^2 + 2 - 2(\xi+\eta) + 2\lambda = 0 \qquad +2\lambda$$
$$\xi^2 - 2\xi + 1 + \eta^2 - 2\eta + 1 = 0$$
$$\underline{(\xi-1)^2 + (\eta-1)^2 + 2\lambda = 0}$$

$$\underline{a^2 + b^2 - 2c = 0}$$

$sG(Q) = (3,1,0)$

$rg(Q) = (2,0,1)$

název: eliptický paraboloid

normální tvar: $a^2+b^2-2c=0$

$$\left.\begin{aligned} a &= \xi-1 \\ b &= \eta-1 \\ c &= -\lambda \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \xi &= a+1 \\ \eta &= b+1 \\ \lambda &= -c \end{aligned}\right\} \Rightarrow \begin{aligned} x &= a+1+b+1 \\ y &= b+1-4c \\ z &= -c \end{aligned}$$

2

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

1. 10 b

$$y''' - \frac{2}{x^2}y' + \frac{4}{x^3}y = 0 \quad \Rightarrow \quad x^3y''' - 2xy' + 4y = 0$$

$$x>0 \;\Rightarrow\; x = e^t \;\Rightarrow\; t = \ln(x) \;\Rightarrow\; \frac{dt}{dx} = \frac{1}{x}$$

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{1}{x}\dot{y}$$

$$y'' = -\frac{1}{x^2}\dot{y} + \frac{1}{x^2}\ddot{y}$$

$$y''' = \frac{2}{x^3}\dot{y} - \frac{1}{x^3}\ddot{y} - \frac{2}{x^3}\ddot{y} + \frac{1}{x^3}\dddot{y} = \frac{2}{x^3}\dot{y} - \frac{3}{x^3}\ddot{y} + \frac{1}{x^3}\dddot{y}$$

$$2\dot{y} - 3\ddot{y} + \dddot{y} - 2\dot{y} + 4y = 0$$

$$\underline{\dddot{y} - 3\ddot{y} + 4y = 0}$$

$$(\lambda^3 - 3\lambda^2 + 4) : (\lambda - 2) = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1)$$

$$-\lambda^2 \qquad -2\lambda + 4$$

← všechny kořeny

$$F_S = \{e^{2t};\ te^{2t};\ e^{-t}\} \Rightarrow F_S^{II} = \{x^2;\ x^2\ln(x);\ \tfrac{1}{x}\}$$

$$\underline{y(t) = Ae^{2t} + Bte^{2t} + Ce^{-t}}$$

$$\underline{y(x) = Ax^2 + Bx^2\ln(x) + \frac{C}{x}}$$

$$y(1) = y'(1) = -1 \;\wedge\; y''(1) = 1$$

$$\left.\begin{aligned} y'(x) &= 2Ax + 2Bx\ln x + Bx - \frac{C}{x^2} \\ y''(x) &= 2A + 2B\ln x + 2B + B + \frac{2C}{x^3} \end{aligned}\right\}$$

$$\left\{\begin{aligned} y(1) &= A + C = -1 \\ y'(1) &= 2A + B - C = -1 \\ y''(1) &= 2A + 3B + 2C = 1 \end{aligned}\right\} \Rightarrow (A, B, C) = (-1, 1, 0)$$

$$\underline{y(x) = -x^2 + x^2\ln x = x^2(\ln x - 1) \qquad x>0}$$

3) $y' - 2\left(\frac{1}{x} + 1\right) y = 0 \quad \Big/ \frac{1}{x^2} e^{-2x}$ $\quad -2\int\left(\frac{1}{x}+1\right)dx = -2\ln x - 2x$

9 bodů

$/F$

$$y' \cdot \frac{1}{x^2} e^{-2x} - 2\left(\frac{1}{x}+1\right)\frac{1}{x^2} e^{-2x} y = 0$$

$$\left(y \cdot \frac{1}{x^2} e^{-2x}\right)' = c'$$

$$y(x) = C \cdot x^2 \cdot e^{2x}$$

$$y(x) = z \cdot x^2 \cdot e^{2x}$$

$$y' = (z'x^2 + 2xz + 2x^2 z) e^{2x}$$

$$y'' = (z''x^2 + 2xz' + 2z + 2xz' + 4xz + 2x^2 z' + 2z'x^2 + 4xz + 4x^2 z) e^{2x}$$

[illegible]!:

$$z''x^4 - x^4 z' = 0$$

$$z'' - z' = 0 \qquad \lambda^2 - \lambda = \lambda(\lambda - 1) = 0$$

$$F_S = \{1, e^x\}$$

$$z(x) = C_1 + C_2 e^x$$

$$y(x) = C_1 x^2 e^{2x} + C_2 x^2 e^{3x}, \qquad I = \mathbb{R}$$

7 bodů

$$A = \begin{pmatrix} 1 & -\alpha & \alpha \\ -\alpha & \alpha & 0 \\ \alpha & 0 & -\alpha^2 \end{pmatrix}$$

$sg(q(\vec{x})) = (1,1,1) \Rightarrow$ alespoň jedno vlastní číslo musí být nulové $\Rightarrow \det(A) \overset{!}{=} 0$

$$\det A = \begin{vmatrix} 1 & -\alpha & \alpha \\ -\alpha & \alpha & 0 \\ \alpha & 0 & -\alpha^2 \end{vmatrix} = -\alpha^3 - \alpha^3 + \alpha^4 \overset{!}{=} 0$$

$$\alpha^4 - 2\alpha^3 \overset{!}{=} 0$$

$$\alpha^3(\alpha-2) \overset{!}{=} 0$$

Dvě možnosti

- $\alpha = 0$ $\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow q(\vec{x}) = x^2$

Tehdy ale $sg(q) = (1,0,2) \Rightarrow$ Nesplňuje to zadání

- $\alpha = 2$ $\quad A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 2 & 0 \\ 2 & 0 & -4 \end{pmatrix}$

Vlastní čísla: $\begin{vmatrix} 1-\lambda & -2 & 2 \\ -2 & 2-\lambda & 0 \\ 2 & 0 & -4-\lambda \end{vmatrix} = (2-3\lambda+\lambda^2)(-4-\lambda) - 4(2-\lambda) - 4(-4-\lambda) =$

$$= -(2-3\lambda+\lambda^2)(4+\lambda) - 8 + 4\lambda + 16 + 4\lambda =$$

$$= -\lambda(\lambda^2+\lambda-2) \overset{!}{=} 0$$

$\alpha)$ $\lambda_1 = 0$ $\qquad \beta)$ $\lambda^2+\lambda-2 = (\lambda+2)(\lambda-1) \overset{!}{=} 0$

$$\lambda_2 = -2 \wedge \lambda_3 = 1$$

$$\Rightarrow sg(q) = (1,1,1)$$

pouze byla-li signatura určena korektně (např. z vlastních čísel)

$$\vec{\sigma}(A) = (1,-2,0) \qquad \sigma(A) = \{1,-2,0\}$$

Numerické chyby u netolerují!

10 bodů

```
In[5]:= Expand[(x - 2 y + z)^2 + (y + 3 z + 1)^2 - 2 (z + 2)]
Out[5]= -3 + x^2 + 2 y - 4 x y + 5 y^2 + 4 z + 2 x z + 2 y z + 10 z^2
In[4]:= Solve[{a == x - 2 y + z, b == y + 3 z + 1, c == z + 2}, {x, y, z}]
Out[4]= {{x → 12 + a + 2 b - 7 c, y → 5 + b - 3 c, z → -2 + c}}
```

$$q(x,y,z) = x^2 - 4xy + 5y^2 + 2xz + 2yz + 10z^2 = (x-2y+z)^2 + 6yz + y^2 + 9z^2 =$$

$$= (x-2y+z)^2 + (y+3z)^2 = \underline{\xi^2 + \eta^2 + 0\lambda^2}$$

$$\left.\begin{aligned} \xi &= x - 2y + z \\ \eta &= y + 3z \\ !!\ \lambda &= z \end{aligned}\right\} \Rightarrow \quad \begin{aligned} x &= \xi + 2\eta - 6\lambda - \lambda = \xi + 2\eta - 7\lambda \\ y &= \eta - 3\lambda \\ z &= \lambda \end{aligned}$$

$$Q(x,y,z) = \xi^2 + \eta^2 - 3 + 2(\eta - 3\lambda) + 4\lambda = \xi^2 + \eta^2 + 2\eta - 2\lambda - 3 = 0$$

$$\underline{\xi^2 + (\eta+1)^2 - 2(\lambda+2) = 0}$$

$$\underline{a^2 + b^2 - 2c = 0}$$

eliptický paraboloid

$SG(Q) = (3,1,0)$

$sg(q) = (2,0,1)$

nemá střed $\Leftarrow$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ nemá řešení

$$\left.\begin{aligned} a &= \xi \\ b &= \eta + 1 \\ c &= \lambda + 2 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \xi &= a \\ \eta &= b - 1 \\ \lambda &= c - 2 \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & -7 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} 12 \\ 5 \\ -2 \end{pmatrix}$$

~~+ definice negativně semidefinitní~~

$$y' + \frac{4x+3y}{9y+6x} = \frac{y}{2x} \qquad \vec{s} = \begin{pmatrix}-3\\2\end{pmatrix}$$

9 bodů

– hledáme nelineární formální řešení $\Rightarrow$ $y(x) = x \cdot w(x)$ $\Rightarrow$ $y' = w + xw'$

$$y' + \frac{4+3\frac{y}{x}}{9\frac{y}{x}+6} = \frac{1}{2}\frac{y}{x} \quad \Rightarrow \quad w + xw' + \frac{4+3w}{9w+6} = \frac{1}{2}w$$

$$xw' = \frac{-4-3w}{9w+6} - \frac{w}{2}$$

$$xw' = \frac{-8-6w-9w^2-6w}{18w+12}$$

$$\frac{18w+12}{9w^2+12w+8}\, w' = -\frac{1}{x}$$

$$\ln|9w^2+12w+8| = -\ln|x| + C$$

$$9w^2+12w+8 = \frac{C}{x}$$

$$9\frac{y^2}{x^2} + 12\frac{y}{x} + 8 = \frac{C}{x}$$

$$9y^2 + 12xy + 8x^2 = Cx$$

$$A = \begin{pmatrix}8&6\\6&9\end{pmatrix} \quad \vec{b} = \begin{pmatrix}+\frac{C}{2}\\0\end{pmatrix} \quad \vec{s} = \begin{pmatrix}-3\\2\end{pmatrix} \Rightarrow \begin{pmatrix}8&6\\6&9\end{pmatrix}\begin{pmatrix}-3\\2\end{pmatrix} \stackrel{!}{=} \begin{pmatrix}-\frac{C}{2}\\0\end{pmatrix}$$

$$\begin{pmatrix}-12\\0\end{pmatrix} \stackrel{!}{=} \begin{pmatrix}-\frac{C}{2}\\0\end{pmatrix}$$

$$8x^2 + 12xy + 9y^2 + 24x = 0$$

!! Hdeši k nuli !!

$$C = -24$$

$$x^2 + \frac{3}{2}xy + \frac{9}{8}y^2 + 3x = 0$$

$$\left(x + \frac{3}{4}y\right)^2 + \frac{9}{16}y^2 + 3x = 0 \quad \Rightarrow$$

~~jde o ... elipsu~~ Jedná se o elipsu

+ její význačné body: a) střed $(-3;2)$

b) průsečík s osou $x=0$: $(0,0)$

c) průsečík s osou $y=0$: $x^2+3x \stackrel{!}{=} 0$ $(-3,0)$ a $(0,0)$

d) průsečík s osou $y=2$:

$$8x^2 + 24x + 36 + 24x \stackrel{!}{=} 0$$

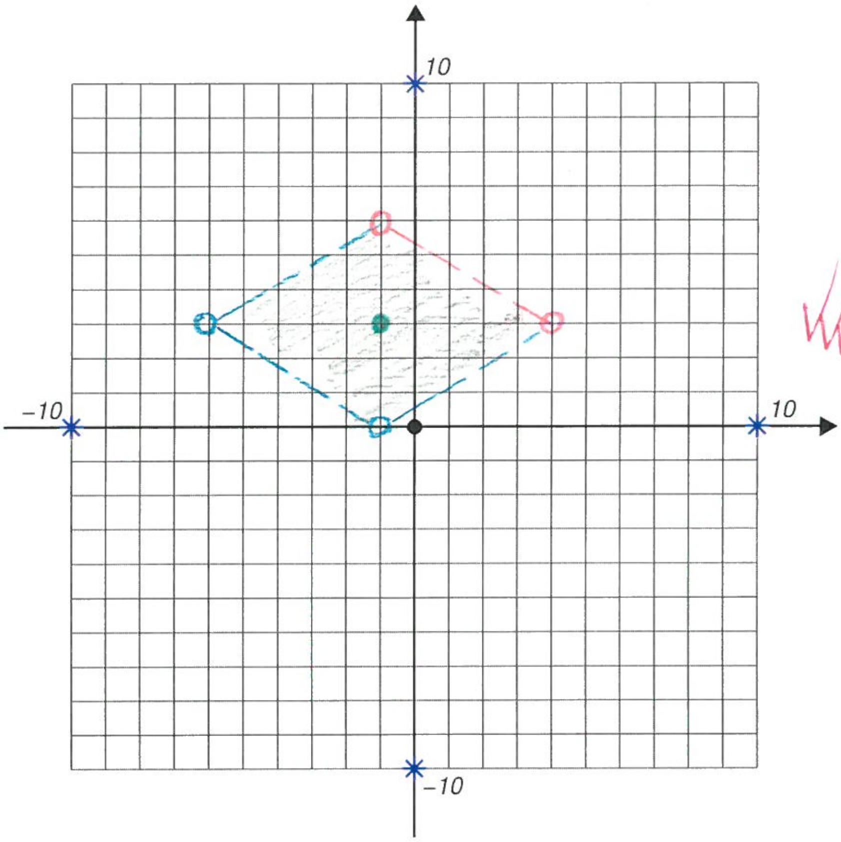
$$x^2 + 6x + \frac{9}{2} \stackrel{!}{=} 0$$

$$x_{1,2} = \frac{1}{2}\left(-6 \pm \sqrt{36-18}\right) = \frac{1}{2}\left(-6 \pm 3\sqrt{2}\right) =$$

$$= -3 \pm \frac{3}{2}\sqrt{2}$$

5,12

5 bodů



za správné detaily v obrázku

$$U_{15}(-1,3) = \left\{\vec{y}\in\mathbb{R}^2 : \rho\left(\vec{y};\binom{-1}{3}\right) < 15\right\}$$

$$\rho\left(\vec{y};\binom{-1}{3}\right) < 15$$

$$3|y_1+1| + 5|y_2-3| < 15$$

Řeším pro $y_1 > -1 \wedge y_2 > 3$ (dále symetrie):

•) hraniční křivka: $3(y_1+1)+5(y_2-3) = 15$

$$3y_1 + 5y_2 = 15 - 3 + 15$$

$$5y_2 = 27 - 3y_1$$

$$y_2 \overset{!}{>} 3 \Rightarrow 27 - 3y_1 > 5\cdot 3 \Rightarrow 3y_1 < 12 \Rightarrow y_1 < 4$$

$$y_1 > -1 \Rightarrow |\, 3y_1 = 27 - 5y_2 \,| \Rightarrow 27 - 5y_2 > -3 \Rightarrow y_2 < 6$$

hraniční křivka $5y_2 = 27 - 3y_2$ je zjevně přímka procházející body $(4,3)$ & $(-1,6)$

Dále symetrie kolem bodu $(-1,3)$

•) $\rho\left(\binom{0}{0}, \binom{-1}{3}\right) = 3|0+1| + 5|0-3| = 3 + 15 = 18$ $\Rightarrow$ NEPATŘÍ!