$y(x) = \sum_{n=0}^{\infty} x^{4n}/4n!$ 8b

$y'(x) = \sum_{n=1}^{\infty} x^{4n-1}/(4n-1)!$! ✓ $\quad y''(x) = \sum_{n=1}^{\infty} x^{4n-2}/(4n-2)!$

$y'''(x) = \sum_{n=1}^{\infty} x^{4n-3}/(4n-3)! \quad y''''(x) = \sum_{n=1}^{\infty} x^{4n-4}/(4n-4)! = \| m = n-1 \| =$

$$= \sum_{m=0}^{\infty} \frac{x^{4m}}{(4m)!} = y(x)$$ ✓

Cauchyova úloha: $\quad y'''' - y = 0$ ✓ $\quad \& \quad y(0)=1 \quad \& \quad y'(0)=0$
$\quad \& \quad y''(0)=0 \quad \& \quad y'''(0)=0$ ✓

Řešení:

$$\lambda^4 - 1 = (\lambda^2-1)(\lambda^2+1) = (\lambda-1)(\lambda+1)(\lambda-i)(\lambda+i) = 0$$

$$\Rightarrow \lambda_1 = 1 \quad \& \quad \lambda_2 = -1 \quad \& \quad \lambda_3 = i \quad \& \quad \lambda_4 = -i$$

$FS = \{e^x, e^{-x}, \cos(x), \sin(x)\}$ ✓

$$y(x) = \alpha e^x + \beta e^{-x} + \mu \cos(x) + \delta \sin(x)$$
$$y'(x) = \alpha e^x - \beta e^{-x} - \mu \sin(x) + \delta \cos(x)$$
$$y''(x) = \alpha e^x + \beta e^{-x} - \mu \cos(x) - \delta \sin(x)$$
$$y'''(x) = \alpha e^x - \beta e^{-x} + \mu \sin(x) - \delta \cos(x)$$

✓ $$\left\{\begin{array}{l} \alpha + \beta + \mu = 1 \\ \alpha - \beta + \delta = 0 \\ \alpha + \beta - \mu = 0 \\ \alpha - \beta + \delta = 0 \end{array}\right\} \Rightarrow (\alpha, \beta, \mu, \delta) = (\tfrac{1}{4}; \tfrac{1}{4}; \tfrac{1}{2}, 0)$$ ✓

$$y(x) = \frac{1}{4}e^x + \frac{1}{4}e^{-x} + \frac{1}{2}\cos(x)$$

$$y(x) = \frac{1}{2}(\cosh(x) + \cos(x)); \quad I = \mathbb{R}$$ ✓

8b

$$y'''' + 2y''' - 2y' - y = 36e^{-x}$$

$$(\lambda^4 + 2\lambda^3 - 2\lambda - 1) : (\lambda - 1) = \lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda+1)^3$$

$3\lambda^3 + 2\lambda - 1$

$3\lambda^2 - 2\lambda - 1$

$\lambda - 1$

$$\Rightarrow \quad FS = \{e^x;\ e^{-x};\ xe^{-x};\ x^2e^{-x}\}$$

Speciální pravá strana: $\quad q(x) = P(x)e^{\alpha x} \qquad \alpha = -1$

je to trojnásobný kořen $\ell(\lambda)$ — konec + příp. konzist.

$$\Rightarrow \text{partik. řešení: } \quad y_p(x) = A \cdot x^3 e^{-x}$$

$(-2)\cdot \quad y_p' = A(3x^2 - x^3)e^{-x}$

$0 \cdot \quad y_p'' = A(6x - 3x^2 - 3x^2 + x^3)e^{-x} = A(6x - 6x^2 + x^3)e^{-x}$

$2 \cdot \quad y_p''' = A(6 - 12x + 3x^2 - 6x + 6x^2 - x^3)e^{-x} = A(6 - 18x + 9x^2 - x^3)e^{-x}$

$1 \cdot \quad y_p'''' = A(-18 + 18x - 3x^2 - 6 + 18x - 9x^2 + x^3)e^{-x} = A(-24 + 36x - 12x^2 + x^3)e^{-x}$

$x^3: \quad 1 - 2 + 2 - 1 = 0$ ✓

$x^2: \quad -12 + 18 - 6 = 0$ ✓

$x: \quad 36 - 36 = 0$

$x^0: \quad A(-24 + 12) \overset{!}{=} 36 \quad \Rightarrow \quad -12A = 36 \quad \Rightarrow \quad A = -3$

$$y(x) = \alpha e^x + \beta e^{-x} + \gamma x e^{-x} + \delta x^2 e^{-x} - 3x^3 e^{-x}; \quad \mathcal{D}om(y) = \mathbb{R}$$

$$\mathcal{S}_0 = [e^x, e^{-x}, xe^{-x}, x^2e^{-x}]_\lambda$$

$$\mathcal{S}_q = [e^x, e^{-x}, xe^{-x}, x^2e^{-x}]_\lambda \underbrace{- 3x^3e^{-x}}_{= y_p(x)}$$

9b

$$x^3y''' + 5x^2y'' - xy' - 8y = 0 \qquad x = e^t \Rightarrow t = \ln x \Rightarrow \frac{dt}{dx} = \frac{1}{x}$$

$$\begin{cases} y' = \dot{y}\frac{1}{x} \\ y'' = \ddot{y}\frac{1}{x^2} - \dot{y}\frac{1}{x^2} \\ y''' = \dddot{y}\frac{1}{x^3} \underbrace{- \frac{2}{x^3}\ddot{y} - \ddot{y}\frac{1}{x^3}}_{-\frac{3}{x^3}\ddot{y}} + \frac{2}{x^3}\dot{y} \end{cases}$$

Вставим:

$$\dddot{y} - 3\ddot{y} + 2\dot{y} + 5\ddot{y} - 5\dot{y} - \dot{y} - 8y = 0$$

$$\dddot{y} + 2\ddot{y} - 4\dot{y} - 8y = 0$$

$$(\lambda^3 + 2\lambda^2 - 4\lambda - 8) : (\lambda - 2) = \lambda^2 + 4\lambda + 4$$

$$\Rightarrow 3 \text{ корня: } \lambda_1 = 2 \text{ и } \lambda_2 = -2 \text{ и } \lambda_3 = -2$$

$$y(t) = Ce^{2t} + De^{-2t} + E \cdot t \cdot e^{-2t}$$

$$y(x) = Cx^2 + \frac{D}{x^2} + \frac{E}{x^2}\ln(x); \quad \mathrm{Dom}(y) = \mathbb{R}^+$$

$$y'(x) = 2Cx - \frac{2D}{x^3} - \frac{2E}{x^3}\ln x + \frac{E}{x^3}$$

$$y''(x) = 2C + \frac{6D}{x^4} + \frac{6E}{x^4}\ln x - \frac{2E}{x^4} - \frac{3E}{x^4}$$

$$\left.\begin{cases} y(1) = C + D \overset{!}{=} -1 \\ y'(1) = 2C - 2D + E \overset{!}{=} 11 \\ y''(1) = 2C + 6D - 2E - 3E \overset{!}{=} -19 \end{cases}\right\} \Rightarrow (C, D, E) = (2, -3, 1)$$

$$y(x) = 2x^2 - \frac{3}{x^2} + \frac{\ln(x)}{x^2}; \quad \mathrm{Dom}(y) = \mathbb{R}^+$$

8b

$$y(x) = x^2 \cdot z(x)$$
$$y' = 2xz + x^2 z'$$
$$y'' = 2z + 2xz' + 2xz' + x^2 z'' = 2z + 4xz' + x^2 z''$$
$$y''' = 2z' + 4z' + 4xz'' + 2xz'' + x^2 z''' = 6z' + 6xz'' + x^2 z'''$$

$$x^3 \cdot (6z' + 6xz'' + x^2 z''') + (4x^3 - 6x^2)(2z + 4xz' + x^2 z') +$$
$$+ (18x - 16x^2 + 4x^3)(2xz + x^2 z') + (-24 + 24x - 8x^2) \cdot x^2 \cdot z = 0$$

$z''' :\ x^5$

$z'' :\ 6x^4 + 4x^5 - 6x^4 = 4x^5$

$z' :\ 6x^3 + 16x^4 - 24x^3 + 18x^3 - 16x^4 + 4x^5 = 4x^5$

$z :\ 8x^3 - 12x^2 + 36x^2 - 32x^3 + 8x^4 - 24x^2 + 24x^3 - 8x^4 = 0$

$$x^5 z''' + 4x^5 z'' + 4x^5 z' = 0$$
$$z''' + 4z'' + 4z' = 0$$
$$\lambda^3 + 4\lambda^2 + 4\lambda = \lambda(\lambda + 4\lambda + 4) = \lambda(\lambda+2)^2$$
$$FS = \{1, e^{-2x}, xe^{-2x}\}$$
$$z(x) = \alpha + \beta e^{-2x} + \gamma x e^{-2x}$$
$$y(x) = \alpha x^2 + \beta x^2 e^{-2x} + \gamma x^3 e^{-2x};\ \mathrm{Dom}(y) = \mathbb{R}$$

5B (7b)

7b

$$x^2 y' + 5y^2 - 2xy = 0 \qquad y(5) = \frac{1}{2}$$

$$y' - \frac{2}{x} y = -\frac{5}{x^2} y^2 \quad / \cdot (-y^{-2})$$

Bernoulli $\alpha = 2 \qquad (1-\alpha) y^{-\alpha} = -y^{-2}$

$$-y^{-2} y' + \frac{2}{x} y^{-1} = \frac{5}{x^2}$$

$$\left.\begin{aligned} z &= y^{-1} \\ z' &= -y^{-2} y' \end{aligned}\right\}$$

$$z' + \frac{2}{x} z = \frac{5}{x^2} \quad / \cdot x^2$$

Integrožni faktor

$p(x) = \frac{2}{x} \qquad P(x) = \ln x^2$

$e^{P(x)} = e^{\ln x^2} = x^2$

$$x^2 z' + 2x z = 5$$

$$(x^2 z)' = (5x + C)'$$

$$x^2 z = 5x + C$$

$$z = \frac{5}{x} + \frac{C}{x^2}$$

$$y = z^{-1} = \left(\frac{5x + C}{x^2}\right)^{-1} = \frac{x^2}{5x + C}$$

poč. podm.

$$\frac{1}{2} = \frac{25}{25 + C} \Rightarrow C = 25$$

$$y(x) = \frac{x^2}{5(x+5)} \qquad I = (-5, +\infty)$$

5B (7b.)

7b

$$x^2 y' + 5y^2 - 2xy = 0 \qquad y(5) = \frac{1}{2}$$

$$y' = \frac{2xy - 5y^2}{x^2} \qquad \text{homogenní}$$

$$y = ux$$
$$y' = u'x + u$$

$$u'x + u = \frac{2u - 5u^2}{1}$$

$$u'x = u - 5u^2$$

$$\frac{u'}{u(1-5u)} = \frac{1}{x}$$

parc. zlomky

$$\int \frac{du}{u(1-5u)} = \ln|x| + C$$

$$\frac{A}{u} + \frac{B}{1-5u} = \frac{A - 5Au + Bu}{u(1-5u)}$$

$$\Rightarrow A = 1 \quad B = 5$$

$$\int \frac{du}{u} + 5\int \frac{du}{1-5u} = \ln|u| - \ln|1-5u| = \ln|x| + C$$

$$\frac{u}{1-5u} = Cx$$

$$\frac{y}{x-5y} = Cx$$

$$y = Cx^2 - 5Cxy$$

$$y = \frac{Cx^2}{1+5Cx}$$

poč. podm.

$$\frac{1}{2} = \frac{C \cdot 25}{1 + C \cdot 25} \Rightarrow C = \frac{1}{25} \qquad y(x) = \frac{x^2}{25 + 5x}$$

$$I = (-5, +\infty)$$