$$\underbrace{u^3 + u^2 - ux^2 + xz - y^3 + 3}_{\triangleq\, G(x,y,z,u)} = 0$$

Nejprve je třeba najít ony 3 generující body tvaru $(2,-1,-4,?)$ 8b

$$u^3 + u^2 - 4u - 8 + 1 + 3 \overset{!}{=} 0$$

$$u^3 + u^2 - 4u - 4 = u^2(u+1) - 4(u+1) = (u^2-4)(u+1) \overset{!}{=} 0$$

3 generující body:

$$\vec{\lambda} = (2,-1,-4,2) \quad \& \quad \vec{\mu} = (2,-1,-4,-2) \quad \& \quad \vec{\varkappa} = (2,-1,-4,-1)$$

Testování nekritičnosti gener. bodů:

$$\frac{\partial G}{\partial u} = 3u^2 + 2u - x^2 \qquad \frac{\partial G}{\partial u}(\vec{\lambda}) = 12 + 4 - 4 = 12 \quad \& \quad \frac{\partial G}{\partial u}(\vec{\mu}) = 4 \quad \& \quad \frac{\partial G}{\partial u}(\vec{\varkappa}) = -3$$

(všechny jsou nekritické)

Pomocné výpočty:

$$\frac{\partial G}{\partial x} = -2ux + z \qquad \frac{\partial G}{\partial y} = -3y^2 \qquad \frac{\partial G}{\partial z} = x \qquad \& \qquad \frac{\partial u}{\partial x} = -\frac{\frac{\partial G}{\partial x}}{\frac{\partial G}{\partial u}}\,;\ \frac{\partial u}{\partial y} = -\frac{\frac{\partial G}{\partial y}}{\frac{\partial G}{\partial u}}\,;\ \frac{\partial u}{\partial z} = -\frac{\frac{\partial G}{\partial y}}{\frac{\partial G}{\partial z}}$$

Tabulka hodnot

	$\vec{\lambda}$	$\vec{\mu}$	$\vec{\varkappa}$
$\partial G/\partial x$	-12	4	0
$\partial G/\partial y$	-3	-3	-3
$\partial G/\partial z$	2	2	2
$\partial u/\partial x$	1	-1	0
$\partial u/\partial y$	$1/4$	$3/4$	-1
$\partial u/\partial z$	$-1/6$	$-1/2$	$2/3$

Výpočet gradientů

$$\operatorname{grad} u^{\vec{\lambda}}(\vec{a}) = (1;\ 1/4;\ -1/6) \ \& \ \operatorname{grad} u^{\vec{\mu}}(\vec{a}) = (-1;\ 3/4;\ -1/2) \ \& \ \operatorname{grad} u^{\vec{\varkappa}}(\vec{a}) = (0;\ -1;\ 2/3)$$

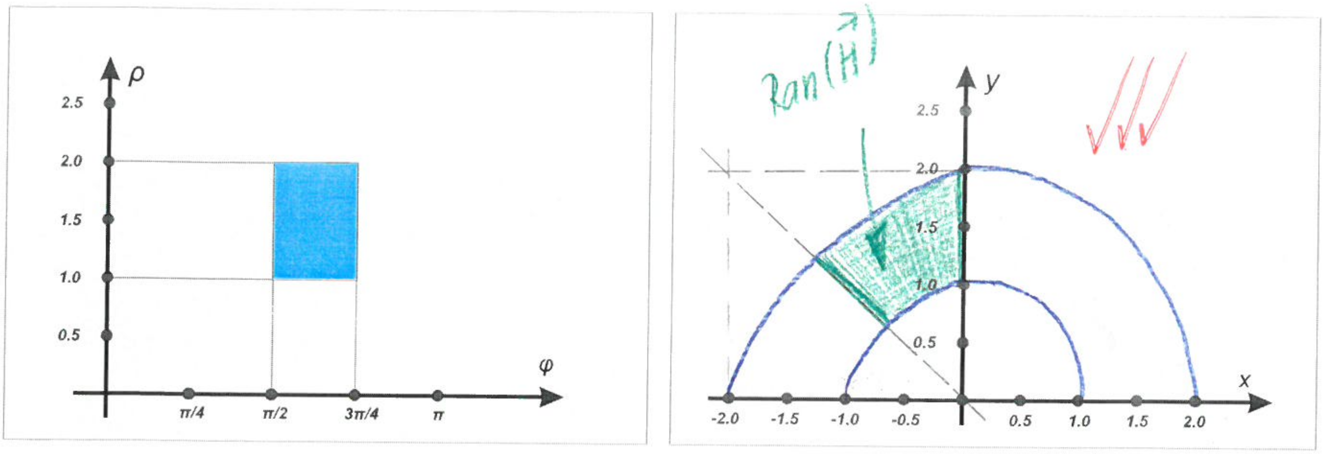
Výpočet směrových derivací

$$\frac{\partial u^{\vec{\lambda}}}{\partial \vec{s}}(\vec{a}) = \frac{1}{\sqrt{3}} \langle \operatorname{grad} u^{\vec{\lambda}}(\vec{a}) \,|\, \vec{s} \rangle = \frac{1}{\sqrt{3}}\left(1 + \frac{1}{4} - \frac{1}{6}\right) = \frac{1}{\sqrt{3}}\,\frac{13}{12}$$

$$\frac{\partial u^{\vec{\mu}}(\vec{a})}{\partial \vec{s}} = \frac{1}{\sqrt{3}}\left(-\frac{3}{4}\right) \quad \& \quad \frac{\partial u^{\vec{\varkappa}}(\vec{a})}{\partial \vec{s}} = \frac{1}{\sqrt{3}}\left(-\frac{1}{3}\right) \quad \& \quad -\frac{3}{4} < -\frac{1}{3}$$

Závěr

Nejstrměji klesá implicitní funkce, jejíž funkční hodnota je rovna -2.



Obrázek k příkladu č. 2

$a = ye^{3x}$
$b = y\cdot e^{-x}$

Testování regularity:

$$\det\left(\frac{D(a,b)}{D(x,y)}\right) = \begin{vmatrix} 3ye^{3x} & e^{3x} \\ -ye^{-x} & e^{-x} \end{vmatrix} = e^{2x}\begin{vmatrix} 3y & 1 \\ -y & 1 \end{vmatrix} = 4ye^{2x}$$

9b

Maximální množina regularity: $G = \{(x,y) \in \mathbb{R}^2 : y \neq 0\}$

regularita ✓

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a} 3ye^{3x} + \frac{\partial u}{\partial b}(-ye^{-x}) \quad \Big| \cdot (-1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial a} e^{3x} + \frac{\partial u}{\partial b} e^{-x} \quad \Big| \cdot 0$$

$$✓\ \frac{\partial^2 u}{\partial x^2} = 9y^2e^{6x}\frac{\partial^2 u}{\partial a^2} + y^2e^{-2x}\frac{\partial^2 u}{\partial b^2} - 6y^2e^{2x}\frac{\partial^2 u}{\partial a\partial b} + 9ye^{3x}\frac{\partial u}{\partial a} + ye^{-x}\frac{\partial u}{\partial b} \quad \Big| \cdot 1$$

$$✓\ \frac{\partial^2 u}{\partial y^2} = e^{6x}\frac{\partial^2 u}{\partial a^2} + e^{-2x}\frac{\partial^2 u}{\partial b^2} + 2e^{2x}\frac{\partial^2 u}{\partial a\partial b} \quad \Big| (-3y^2)$$

$$✓\ \frac{\partial^2 u}{\partial x\partial y} = 3ye^{6x}\frac{\partial^2 u}{\partial a^2} - ye^{-2x}\frac{\partial^2 u}{\partial b^2} + 2ye^{2x}\frac{\partial^2 u}{\partial a\partial b} + 3e^{3x}\frac{\partial u}{\partial a} - e^{-x}\frac{\partial u}{\partial b} \quad \Big| \cdot (-2y)$$

$$0\cdot\frac{\partial^2 u}{\partial a^2} + 0\cdot\frac{\partial^2 u}{\partial b^2} - 16y^2\,\overset{e^{2x}}{\downarrow}\,\frac{\partial^2 u}{\partial a\partial b} + 0\cdot\frac{\partial u}{\partial a} + 4ye^{-x}\frac{\partial u}{\partial b} = 0 \quad \Big/ \begin{matrix} y \neq 0 \\ e^{-x} \neq 0 \end{matrix}$$

$$4ye^{3x}\frac{\partial^2 u}{\partial a\partial b} - \frac{\partial u}{\partial b} = 0 \ ✓$$

$$4a\frac{\partial^2 u}{\partial a\partial b} - \frac{\partial u}{\partial b} = 0 \ ✓$$

Řešení:

$$v(a,b) \stackrel{\Delta}{=} \frac{\partial u}{\partial b}(a,b)$$

✓

$$4a\frac{\partial v}{\partial a} - v = 0$$

$$\frac{\partial v}{\partial a} - \frac{1}{4a}v = 0 \quad \Big| \ a^{-1/4} \leftarrow \text{integrační faktor}$$

$$(v\cdot a^{-1/4})' = 0$$

$$v(a,b) = \omega(b)\cdot a^{1/4} \ ✓$$

$$u(a,b) = \varkappa(b)\cdot a^{1/4} + \eta(a)$$

$$u(x,y) = \varkappa(y\cdot e^{-x})\cdot(ye^{3x})^{1/4} + \eta(ye^{3x})$$

✓ (stačí jedno z toho)

$z^3 - 2xz^2 + 6y = 30 \qquad (x_0, y_0) = (3,5)$ 8b

Nejprve je třeba najít generující bod. Ten bude tvaru $(x_0, y_0, z_0) = (3, 5, ?)$

$$z^3 - 6z^2 + 30 = 30 \quad \Rightarrow \quad z^2(z-6) = 0$$

Existují 2 generující body:

$$\vec{\mu} = (3,5,0) \quad \& \quad \vec{\lambda} = (3,5,6)$$

$$z_0 = 0 \qquad\qquad z_0 = 6$$

Ověřím existenční podmínky:

$$\frac{\partial G}{\partial z} = \frac{\partial}{\partial z}\left(z^3 - 2xz^2 + 6y - 30\right) = 3z^2 - 4xz$$

$$\frac{\partial G}{\partial z}(\vec{\mu}) = 0 \quad \Rightarrow \quad \vec{\mu} \text{ je kritickým bodem } (\Rightarrow \text{dále nevyšetřujeme})$$

$$\frac{\partial G}{\partial z}(\vec{\lambda}) = 3\cdot 36 - 12\cdot 6 = 36 \neq 0 \quad \Rightarrow \quad \vec{\lambda} \text{ je OK}$$

Dále vyšetřujeme pouze $\vec{\lambda} = (3,5,6)$

$$\frac{\partial G}{\partial x} = -2z^2 \qquad \frac{\partial G}{\partial x}(\vec{\lambda}) = -72 \qquad \frac{\partial z}{\partial x} = -\frac{\frac{\partial G}{\partial x}(\vec{\lambda})}{\frac{\partial G}{\partial z}(\vec{\lambda})} = -\frac{-72}{36} = 2$$

$$\frac{\partial G}{\partial y} = 6 \qquad \frac{\partial G}{\partial y}(\vec{\lambda}) = 6 \qquad \frac{\partial z}{\partial y} = -\frac{\partial G/\partial y(\vec{\lambda})}{\frac{\partial G}{\partial z}(\vec{\lambda})} = -\frac{6}{36} = -\frac{1}{6}$$

$$dz_{(3,5)}(dx, dy) = 2\,dx - \frac{1}{6}\,dy = 2(x-3) - \frac{1}{6}(y-5)$$

Rovnice tečné roviny:

$$z - 6 = 2(x-3) - \frac{1}{6}(y-5)$$

$$6z - 36 = 12x - 36 - y + 5$$

$$12x - y - 6z + 5 = 0$$

Obecný předpis přímek (procházejících bodem $\vec{0}$):

$$M_a = \{(x,y) \in R^2 : y = ax\}$$

Hledáme $a \in R$ tak, aby $\lim\limits_{\substack{(x,y)\to\vec{0} \\ (x,y)\in M_a}} g(x) \overset{!}{=} \frac{16}{5}$

6b

formulace metody řešení

Výpočet č. 1

$$\lim_{\substack{(x,y)\to\vec{0} \\ (x,y)\in M_a}} g(x,y) = \lim_{x\to 0} g(x, a\cdot x) = \lim_{x\to 0} \frac{a^2x^2}{x^2\sqrt{9+a^2}} = \frac{a^2}{\sqrt{9+a^2}} \overset{!}{=} \frac{16}{5}$$

$$5a^2 = 16\sqrt{9+a^2}$$

$$25a^4 - 16^2 \cdot a^2 - 16^2 \cdot 9 = 0$$

$$(a^2)_{1,2} = \frac{1}{50}\left(16^2 \pm \sqrt{16^4 + 4 \cdot 25 \cdot 16^2 \cdot 9}\right) \Rightarrow \text{minus nás nezajímá}$$

$$a^2 = \frac{16}{50}\left(16 + 2\sqrt{64+225}\right) = \frac{16}{50}(16 + 2\cdot 17) = 16$$

$$a = \pm 2^2 = \pm 4$$

1. odpověď: $y = 4x$ & $y = -4x$ ← stačí jedno z těchto řešení

Směrové vektory těchto přímek: $\vec{s} = (1,4)$ & $\vec{r} = (-1,4)$ ← stanovení směrového vektoru

$$\frac{\partial g}{\partial \vec{s}}(0,0) = \frac{1}{\sqrt{17}} \lim_{\tau\to 0} \frac{g(\tau; 4\tau) - g(0,0)}{\tau} = \frac{1}{\sqrt{17}} \lim_{\tau\to 0} \frac{1}{\tau}\left(\frac{16\tau^2}{|\tau|\cdot\sqrt{9\tau^2+16\tau^2}} - \frac{16}{5}\right)$$

$$= \lim_{\tau\to 0} \frac{1}{\sqrt{17}}\left(\frac{16}{\sqrt{9+16}} - \frac{16}{5}\right) = \lim_{\tau\to 0} \frac{1}{\sqrt{17}} \cdot 0 = 0$$

Analogicky:

$$\frac{\partial g}{\partial \vec{r}}(0,0) = 0$$

$+10$

$$\frac{\partial g}{\partial x} = 2x - 2y + 2z \overset{!}{=} 0$$
$$\frac{\partial g}{\partial y} = -2x + 4z + 18 \overset{!}{=} 0$$
$$\frac{\partial g}{\partial z} = 2x + 4y - 10 \overset{!}{=} 0$$

$\operatorname{grad} g \overset{!}{=} 0$

6b

$$\Rightarrow \begin{cases} x - y + z = -5 \\ x - 2z = 9 \\ x + 2y = 5 \end{cases} \Rightarrow (x, y, z) = (1, 2, -4)$$

Stacionární bod: $\vec{a} = (1, 2, -4)$

(je-li stacionární bod určen chybně, dále nelze získávat body!)

$$\frac{\partial^2 g}{\partial x^2} = 2 \quad \frac{\partial^2 g}{\partial y^2} = 0 \quad \frac{\partial^2 g}{\partial z^2} = 0 \quad \frac{\partial^2 g}{\partial x \partial y} = -2 \quad \frac{\partial^2 g}{\partial x \partial z} = 2 \quad \frac{\partial^2 g}{\partial y \partial z} = 4$$

$$H|_{\vec{a}} = \begin{pmatrix} 2 & -2 & 2 \\ -2 & 0 & 4 \\ 2 & 4 & 0 \end{pmatrix} \dots$$ Hessova matice ve stacionárním bodě

nemusí být uvedeno ve formátu matice

Polovina druhého totálního diferenciálu:

$$\frac{1}{2} d^2 g_{\vec{a}}(dx, dy) = dx^2 - 2dxdy + 2dxdz + 4dydz =$$
$$= (dx - dy + dz)^2 - dy^2 - dz^2 + 6dydz =$$
$$= \underbrace{(dx - dy + dz)^2 - (dy - 3dz)^2 + 8dz^2}_{\text{doplnění na čtverce}} \lessgtr 0$$

stanovení typu definitnosti

Bod $\vec{a}$ je sedlovým bodem, tj. stacionárním bodem, v němž extrém nenastává.

musí být ale určen korektně!