$$I \triangleq I(b) = \int_0^{+\infty} \frac{e^{-ax^2}}{x}\left[\sin(bx^2) - \sin(ax^2)\right] dx$$

Tři předpoklady:

- $I(b=c) = 0$ ✓
- integrand měřitelný, tj. $f(x|a,b,c) \in \Lambda_x$
- $\left|\frac{\partial f}{\partial b}(x|a,b,c)\right| = \left|\frac{e^{-ax^2}}{x}\cdot\cos(bx^2)\cdot x^2\right| =$

$$= \left|e^{-ax^2}\cdot x \cdot \cos(bx^2)\right| \leq x\cdot e^{-ax^2} \in \mathcal{L}(0,+\infty)$$

$$\int_0^{\infty} x\cdot e^{-ax^2} dx = \left|\begin{matrix} u = x^2 \\ du = 2x\,dx\end{matrix}\right| = \frac{1}{2}\int_0^{\infty} e^{-au} du = \frac{1}{2a} \in \mathbb{R}$$

✓

Lze derivovat:

$$\frac{dI}{db} = \int_0^{\infty} x\cdot e^{-ax^2}\cos(bx^2)\,dx = \left\|\begin{matrix} u = x^2 \\ du = 2x\,dx\end{matrix}\right| = \frac{1}{2}\int_0^{\infty} e^{-au}\cdot\cos(bu)\,du =$$

$$= \frac{a}{a^2+b^2}\,\frac{1}{2} \quad \leftarrow \text{viz} \quad \int_0^{\infty} e^{-ax} e^{i\omega x} dx = \frac{a+i\cdot\omega}{a^2+\omega^2}$$

viz komplexní integrál

$$I(b) = \frac{1}{2}\int \frac{a}{a^2+b^2} db = \frac{1}{2a}\int \frac{1}{1+(b/a)^2} db = \frac{1}{2}\operatorname{arctg}\frac{b}{a} + C$$

$$I(b=c) = \frac{1}{2}\operatorname{arctg}\frac{c}{a} + C \overset{!}{=} 0 \Rightarrow C = -\frac{1}{2}\operatorname{arctg}\frac{c}{a}$$

$$\int_0^{\infty} \frac{e^{-ax^2}}{x}\left[\sin(bx^2) - \sin(cx^2)\right] dx = \frac{1}{2}\operatorname{arctg}\frac{b}{a} - \frac{1}{2}\operatorname{arctg}\frac{c}{a}$$

4 (8 + 1 navíc bodů)

Nechť je Lebesgueova míra zadána prostřednictvím vytvořující funkce $\varphi(\tau) = \tau^2 \operatorname{sgn}(\tau)$ platné pro obě dimenze. Nalezněte těžiště množiny

$$X = \left\{ (x, y) \in \mathbf{E}^2 : y > 0 \wedge \sqrt{\left|\frac{x}{a}\right|} + \sqrt{\frac{y}{b}} \leqslant 1 \right\}.$$

← lze použít jen v 1. kvadrantu

$$\left.\begin{aligned} x &= a\rho \cos^4 \varphi \\ y &= b\rho \sin^4(\varphi) \end{aligned}\right\} \Rightarrow \det\left(\frac{D(x,y)}{D(\rho,\varphi)}\right) = 4ab\rho \cos^3\varphi \cdot \sin^3\varphi$$

$$\mu_2(X) = \int_X 1 \, d\mu(x) = \int_X |2x| \cdot |2y| \, d(x,y) = 4 \int_X |xy| \, d(x,y)$$

$$x_T = \frac{1}{\mu_2(X)} \int_X x \, d\mu(x,y) = \frac{4}{\mu_2(X)} \int_X x \cdot |xy| \, d(x,y) = 0$$ ← musí být z důvodu symetrie

$$y_T = \frac{4}{\mu_2(X)} \int_X |x| \cdot y^2 \, d(x,y)$$

a)

$$\mu_2(X) = 8 \int_{\substack{X \\ x>0}} a\rho \cos^4 \cdots \sin^4(\varphi) \cdot 4ab\rho \cos^3(\varphi) \cdot \sin^3 \cdots \, d(\rho,\varphi) =$$

$$= 32 a^2 b^2 \int_0^1 \rho^3 \, d\rho \cdot \int_0^{\pi/2} \cos^7(\varphi) \cdot \sin^7(\varphi) \, d\varphi = 8a^2b^2 \frac{\Gamma(4) \cdot \Gamma(4)}{\Gamma(8)} = 8a^2b^2 \frac{6 \cdot 6}{2 \cdot 7!} =$$

$$= \frac{1}{35} a^2 b^2$$

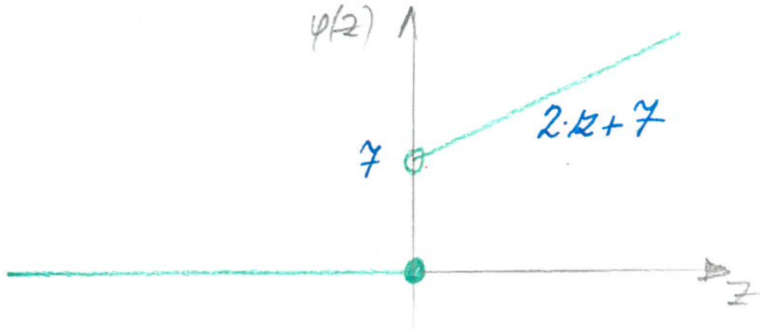
b)

$$y_T = \frac{4 \cdot 35}{a^2 b^2} \cdot 2 \int_{\substack{X \\ x>0}} 4 a^2 b^3 \rho^4 \cos^7(\varphi) \cdot \sin^{11}(\varphi) \, d(\rho,\varphi) = \frac{8 \cdot 4 \cdot 35}{a^2 b^2} a^2 b^3 \int_0^1 \rho^4 \, d\rho \cdot \int_0^{\pi/2} \cdots$$

$$= 8 \cdot 4 \cdot 35 \, b \cdot \frac{1}{5} \frac{\Gamma(4) \cdot \Gamma(6)}{2 \cdot \Gamma(10)} = 4 \cdot 4 \cdots b \frac{6 \cdot 5!}{9!} = \frac{4 \cdot 4}{9 \cdot 8} = \frac{2}{9} b$$

← za úplné řešení (souřadnice) těžiště $y_T = \frac{2}{9} b$

$$\varphi(z) = (2z+7)\cdot\Theta(z)$$



$$\mu(A) = \| A \in \mathcal{H}_1 \| = \varphi(3) - \varphi(0) = 2\cdot 3 + 7 - 0 = 13$$

$$B = (0,3) = \bigoplus_{n=1}^{\infty} \underbrace{\left\langle \frac{3}{n+1}; \frac{3}{n}\right)}_{\in \mathcal{H}_1}$$

$$\mu(B) = \sum_{n=1}^{\infty} \mu\left(\left\langle \frac{3}{n+1}; \frac{3}{n}\right)\right) = \sum_{n=1}^{\infty}\left[\varphi\left(\frac{3}{n}\right) - \varphi\left(\frac{3}{n+1}\right)\right] =$$

σ-aditivita

$$= \left\| s_n = \sum_{k=1}^{n}\left[\varphi\left(\frac{3}{k}\right) - \varphi\left(\frac{3}{k+1}\right)\right] = \varphi(3) - \varphi\left(\frac{3}{n+1}\right) \right\| =$$

$$= \lim_{n\to+\infty}\left[\varphi(3) - \varphi\left(\frac{3}{n+1}\right)\right] = \varphi(3) - \lim_{n\to+\infty}\varphi\left(\frac{3}{n+1}\right) =$$

$$= \varphi(3) - \lim_{z\to 0_+}\varphi(z) = 2\cdot 3 + 7 - 7 = 6$$

odrazem

$x = a\rho \cos^2\vartheta \cos^2\varphi$
$y = b\rho \cos^2\vartheta \sin^2\varphi$
$z = c\rho \sin^2\vartheta$

$$\left(\frac{x}{a} + \frac{y}{b}\right)^2 + \frac{z^2}{c^2} < \frac{x}{a} \cdot \frac{z}{c}$$

9 b.

$$\rho^2\cos^4\vartheta + \rho^2\sin^4\vartheta < \rho\cos^2\vartheta$$

$$\rho < \cos^2\vartheta$$

$$x, y, z > 0 \Rightarrow \varphi \in (0, \pi/2) \wedge \underline{\vartheta \in (0, \pi/2)}$$
$$\cos\vartheta > 0$$

$$\mu_3(T) = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\cos^2\vartheta} \overbrace{2abc\,\rho^2 \cos^3\vartheta \sin\vartheta \cos\varphi \sin\varphi}^{\text{jacob.}} \, d\rho \, d\vartheta \, d\varphi$$

je-li více při integraci podle ρ 3 patrné, dál se neopakuje

$$= \frac{2}{3} abc \int_0^{\pi/2}\int_0^{\pi/2} \cos^4\vartheta \cdot \cos\varphi \cdot \sin\varphi \, d\vartheta \, d\varphi = \left| \text{z Fubiniho} \right| =$$

$$= \frac{2}{3} abc \cdot \int_0^{\pi/2} \cos^4\vartheta \, d\vartheta \cdot \int_0^{\pi/2} \cos\varphi \cdot \sin\varphi \, d\varphi = \frac{2}{3} abc \cdot \frac{\Gamma(\frac{5}{2}) \cdot \Gamma(\frac{1}{2})}{2\Gamma(3)} \cdot \frac{\Gamma(1) \cdot \Gamma(1)}{2\Gamma(2)} =$$

$$= \frac{2}{3} abc \, \frac{\pi \cdot \frac{3}{4}}{2 \cdot 2} \cdot \frac{1 \cdot 1}{2 \cdot 1} = \frac{2}{3} \pi abc \cdot \frac{3}{16} \cdot \frac{1}{2} = \frac{1}{16} \pi abc$$

$$\alpha = \int_T y \, d(x,y,z) = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\cos^2\vartheta} 2abc\,\rho^2 \cos^3\vartheta \sin\vartheta \cos\varphi \sin\varphi \cdot b\rho \cos^2\vartheta \sin^2\varphi \, d\rho \, d\vartheta \, d\varphi =$$

$$= \frac{1}{2} ab^2c \int_0^{\pi/2}\int_0^{\pi/2} \cos^6\vartheta \cdot \cos\varphi \cdot \sin^3\varphi \, d\vartheta \, d\varphi = \left| \text{z Fubiniho} \right| =$$

$$= \frac{1}{2} ab^2c \int_0^{\pi/2} \cos^6\vartheta \, d\vartheta \cdot \int_0^{\pi/2} \cos\varphi \sin^3\varphi \, d\varphi = \frac{1}{2} ab^2c \, \frac{\Gamma(\frac{7}{2}) \cdot \Gamma(\frac{1}{2})}{2 \cdot \Gamma(4)} \cdot \frac{\Gamma(1) \cdot \Gamma(2)}{2 \cdot \Gamma(3)} =$$

$$= \frac{1}{2} ab^2c \, \pi \, \frac{\frac{1}{8} \cdot 5 \cdot 3 \cdot 1}{3 \cdot 2} \cdot \frac{1}{2} = \frac{5}{16} \cdot \frac{1}{16} \pi ab^2c$$

$$y_T = \frac{\alpha}{\mu_3(T)} = \frac{\frac{5}{16} \cdot \frac{1}{16} \pi ab^2c}{\frac{1}{16} \pi abc} = \frac{5}{16} b$$

(Dá se čekat přímý výsledek, výsledek lze určit bez námahy!)

$\vec{G}(x,y) = (4x; 5x + y^2)$

8b

$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = 5 - 0 = 5$$

$$\left.\begin{aligned} x &= a\rho\sqrt{\cos\varphi} \\ y &= b\rho\sqrt{\sin\varphi} \end{aligned}\right\} \quad \Delta_J = \frac{1}{2}\rho \frac{ab}{\sqrt{\cos\varphi\sin\varphi}} \Rightarrow$$

φ může být pouze v intervalu $(0; \frac{1}{2}\pi)$

$\Rightarrow$ v II., III., IV. kvadrantu užijeme symetrie

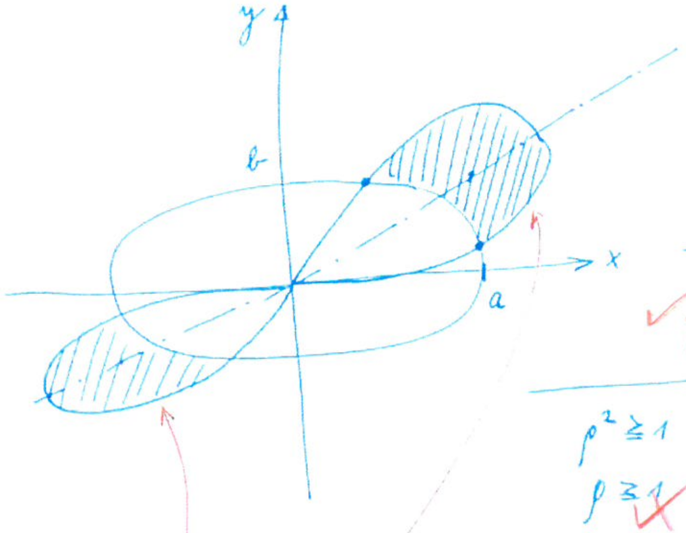
citace věty

$$\int_C \vec{G}(x,y)\, ds_C(x,y) = \left\| \text{Greenova věta} \right\| = 5 \iint_S d(x,y) = \left\| S = \left\{ \frac{x^4}{a^4} + \frac{y^4}{b^4} \leq \omega x y \right\} \right\| =$$

$$= \left\| \begin{aligned} &\frac{x^4}{a^4} + \frac{y^4}{b^4} \leq \omega x y \\ &\rho^4 \leq ab\omega\sqrt{\cos\varphi\sin\varphi}\cdot\rho^2 \end{aligned} \qquad x\cdot y \overset{!}{\geq} 0 \Rightarrow \varphi \in (0; \tfrac{\pi}{2}) \cup (\pi; \tfrac{3\pi}{2}) \right\| =$$

$$= 5\cdot 2 \int_0^{\pi/2} \int_0^{\sqrt{ab\omega}\cdot(\cos\varphi\cdot\sin\varphi)^{1/4}} \frac{1}{2}\rho \frac{ab}{\sqrt{\cos\varphi\sin\varphi}}\, d\rho\, d\varphi = \frac{5}{2}ab \int_0^{\pi/2} \left[\rho^2\right]_0^{\sqrt{ab\omega}\cdot(\cos\varphi\sin\varphi)^{1/4}} \frac{1}{\sqrt{\cos\varphi\sin\varphi}}\, d\varphi =$$

$$= \frac{5}{2}ab\cdot a\cdot b\cdot\omega \int_0^{\pi/2} 1\, d\varphi = \frac{5}{2}a^2b^2\omega\cdot\frac{\pi}{2} = \frac{5}{4}\pi\omega a^2 b^2$$



4

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} \geqq 1$$

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2 \leqq 4\frac{x}{a}\frac{y}{b}$$

$$\left\{ \begin{array}{l} x = a\rho\cos\varphi \\ y = b\rho\sin\varphi \end{array} \right\} \Rightarrow dx\,dy = ab\rho\,d\rho\,d\varphi$$

$$\rho^2 \geqq 1 \;\wedge\; \rho^4 \leqq 4\rho^2\cos\varphi\sin\varphi$$

$$\rho \geqq 1 \;\wedge\; \rho \leqq \sqrt{2}\sqrt{\sin 2\varphi}$$

Body dotyku obou křivek: $\rho = 1 \Rightarrow 1 = \sqrt{2}\sqrt{\sin 2\varphi}$

$$\sin(2\varphi) = \frac{1}{2}$$

určení mezí

$$\varphi_{min} = \frac{\pi}{12} \qquad \varphi_{max} = \frac{5\pi}{12}$$

nejsou-li meze určeny správně, dále se neopravuje!

$$\mu_2(S) = 2\cdot\int_{\pi/12}^{5\pi/12}\int_{1}^{\sqrt{2\cdot\sin 2\varphi}} ab\rho\,d\rho\,d\varphi = 2\cdot\frac{1}{2}\int_{\pi/12}^{5\pi/12} ab\left[\rho^2\right]_1^{\sqrt{2\cdot\sin 2\varphi}} d\varphi =$$

$$= ab\int_{\pi/12}^{5\pi/12}\left(2\cdot\sin(2\varphi)-1\right)d\varphi = 2ab\int_{\pi/12}^{5\pi/12}\sin(2\varphi)\,d\varphi - ab\frac{\pi}{3} =$$

$$= 2ab\int_{\pi/12}^{5\pi/12}\sin(2\varphi)\,d\varphi - \frac{\pi}{3}ab = ab\left[-\cos(2\varphi)\right]_{\pi/12}^{5\pi/12} - \frac{\pi}{3}ab =$$

$$= ab\left(\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}\right) - \frac{\pi}{3}ab = ab\left(\sqrt{3}-\frac{\pi}{3}\right)$$

$$x^2+y^2+4(x+y)+4 = (x+2)^2+(y+2)^2-4-4+4=0$$

$$(x+2)^2+(y+2)^2=4$$

9b

$$\left.\begin{array}{l} x=-2+2\cos t \\ y=-2+2\sin t \end{array}\right\} \quad t\in\langle 0;2\pi)$$

$$\dot{\vec{\varphi}}(t)=(-2\sin t;\ 2\cos t) \qquad \& \qquad \|\dot{\vec{\varphi}}(t)\|=2$$

$$x^2+(y+2)^2 = 4+8\cos t+4\cos^2 t+4\sin^2 t = 8+8\cos t = 8(1+\cos t)$$

$$\int_B \sqrt{x^2+(y+2)^2}\, d\mu_c(x,y) = \int_0^{2\pi} \sqrt{8}\sqrt{1+\cos t}\cdot 2\, dt = 4\sqrt{2}\int_0^{2\pi}\sqrt{1+\cos t}\, dt =$$

$$= 4\sqrt{2}\int_0^{2\pi} \frac{\sqrt{1+\cos t}\cdot\sqrt{1+\cos t}}{\sqrt{1+\cos t}}\, dt = 4\sqrt{2}\int_0^{2\pi}\frac{|\sin t|}{\sqrt{1+\cos t}}\, dt =$$

$$= 4\sqrt{2}\int_0^{\pi}\frac{\sin t}{\sqrt{1+\cos t}}\, dt - 4\sqrt{2}\int_{\pi}^{2\pi}\frac{\sin t}{\sqrt{1+\cos t}}\, dt = \left\| \begin{array}{l} u=1+\cos t \\ du=-\sin t\, dt \end{array} \right\| =$$

$$= +4\sqrt{2}\int_0^2 \frac{1}{\sqrt{u}}\, du \mp 4\sqrt{2}\int_2^0 \frac{1}{\sqrt{u}}\, du = +8\sqrt{2}\int_0^2\frac{1}{\sqrt{u}}\, du = +8\sqrt{2}\left[2\sqrt{u}\right]_0^2 =$$

32

$$= 8\cdot\sqrt{2}\cdot 2\cdot\sqrt{2} = \underline{\underline{32}}$$

$$\int_0^{2\pi}\sqrt{1+\cos t}\, dt = 4\sqrt{2}$$

$$\int_0^{2\pi} \sqrt{1-\cos t}$$

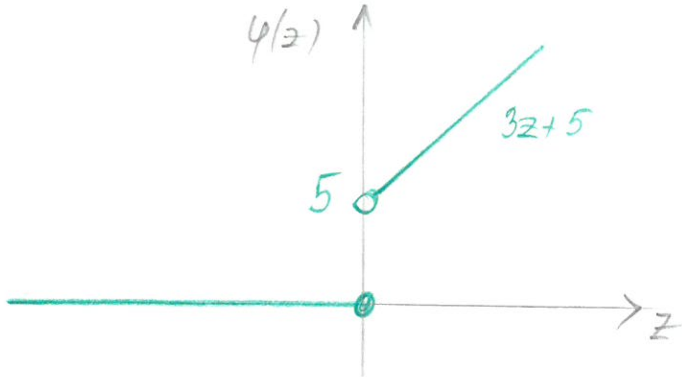
$$\int_0^{2\pi} \frac{|\sin t|}{1+\cos t}$$

$$\int_0^{\pi} \frac{\sin t}{\sqrt{1+\cos t}} - \int_{\pi}^{2\pi} \frac{\sin t}{\sqrt{1+\cos t}} \qquad \left| \begin{array}{l} u = 1+\cos t \\ \frac{du}{dt} = -\sin t \end{array} \right|$$

$$-\int_2^0 \frac{1}{\sqrt{u}} + \int_0^2 \frac{1}{\sqrt{u}}$$

$$= 2\int_0^2 \frac{1}{\sqrt{u}} \quad \checkmark$$

$\varphi(z) = (3z+5)\cdot\Theta(z)$ 5b



•) $\mu(A) = \varphi(2) - \varphi(0) = 11 - 0 = 11$

$\Leftarrow \quad A \in \mathcal{H}_1$

••) $B \in \mathcal{H}_2 \qquad (0, \alpha) = \biguplus_{n=1}^{\infty} \underbrace{\langle \frac{\alpha}{n+1} ; \frac{\alpha}{n})}_{\in \mathcal{H}_1}$

$$\mu(B) = \sum_{n=1}^{\infty} \mu\left(\left\langle \frac{\alpha}{n+1} ; \frac{\alpha}{n} \right)\right) = \| \alpha = 2 \| = \sum_{n=1}^{\infty} \mu\left(\left\langle \frac{2}{n+1} ; \frac{2}{n} \right)\right) =$$

$$= \sum_{n=1}^{\infty} \left[\varphi\left(\frac{2}{n}\right) - \varphi\left(\frac{2}{n+1}\right) \right] = \left\| \begin{array}{l} s_n = \sum_{k=1}^{n} \left[\varphi\left(\frac{2}{k}\right) - \varphi\left(\frac{2}{k+1}\right) \right] = \\ = \varphi(2) - \varphi\left(\frac{2}{n+1}\right) \end{array} \right\| =$$

$$= \lim_{n\to\infty} \left[\varphi(2) - \varphi\left(\frac{2}{n+1}\right) \right] = \varphi(2) - \lim_{n\to\infty} \varphi\left(\frac{2}{n+1}\right) =$$

$$= \varphi(2) - \lim_{z \to 0_+} \varphi(z) = \varphi(2) - \varphi(0_+) = 11 - 5 = 6$$

VW odvozem

$$I(b) = \int_0^\infty e^{-ax^2} \frac{\cos(bx^2) - \cos(cx^2)}{x} \, dx$$

9b

Tři předpoklady:

- $I(b=c) = 0$
- $f(x|a,b,c) \in \mathcal{R}_x$
- $\left|\frac{\partial f}{\partial b}(x,a,b,c)\right| = \left|-x \cdot e^{-ax^2} \sin(bx)\right| \leq x \cdot e^{-ax^2} \in \mathcal{L}(0,+\infty)$

$$\underbrace{\int_0^\infty x \cdot e^{-ax^2} dx = \left\| \begin{matrix} u = x^2 \\ du = 2x\,dx \end{matrix} \right\| = \frac{1}{2} \int_0^\infty e^{-au} du = \frac{1}{2a} \in \mathbb{R}}$$

Lze derivovat podle b:

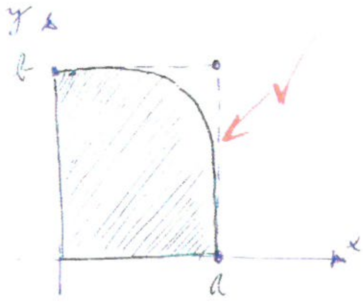
$$\frac{dI}{db} = -\int_0^\infty x \cdot e^{-ax^2} \sin(bx)\, dx = \left\| \begin{matrix} u = x^2 \\ du = 2x\,dx \end{matrix} \right\| = \frac{1}{2} \int_0^\infty e^{-au} \sin(bu)\, du =$$

$$= -\frac{1}{2} \frac{b}{a^2+b^2}$$

$$I(b) = -\frac{1}{2} \int \frac{b}{a^2+b^2} db = -\frac{1}{4} \ln(a^2+b^2) + d$$

$$I(b=c) = -\frac{1}{4} \ln(a^2+c^2) + d \stackrel{!}{=} 0 \quad \Rightarrow \quad d = \frac{1}{4} \ln(a^2+c^2)$$

$$\int_0^\infty e^{-ax^2} \frac{\cos(bx^2) - \cos(cx^2)}{x} dx = \frac{1}{4} \ln \frac{a^2+c^2}{a^2+b^2}$$



$$\frac{x^4}{a^4} + \frac{y^4}{b^4} \le 1$$

Jb

$$\left.\begin{aligned} x &= a\rho\sqrt{\cos\varphi} \\ y &= b\rho\sqrt{\sin\varphi} \end{aligned}\right\}$$

$$\det\left(\frac{D(x,y)}{D(\rho,\varphi)}\right) = \frac{1}{2} a b \rho \frac{1}{\sqrt{\cos\varphi}} \frac{1}{\sqrt{\sin\varphi}}$$

$$\mu_2(O) = (\varphi(a) - \varphi(0)) \cdot (\psi(b) - \psi(0)) = a^4 b^4$$

$$\mu_2(W) = \int_W 1 \, d\mu_2(x,y) = \int_W 4x^3 4y^3 \, d\lambda(x,y) = 16 \int_W x^3 y^3 \, d(x,y) =$$

$$= \left| \text{substituce} \right| = 16 \int_0^{\pi/2} \int_0^1 a^3 \rho^3 \cos^{3/2}(\varphi)\, b^3 \rho^3 \sin^{3/2}(\varphi) \cdot \frac{1}{2} a b \rho \frac{1}{\sqrt{\cos\varphi}} \frac{1}{\sqrt{\sin\varphi}} \, d\rho \, d\varphi =$$

$$= \left| \begin{matrix} \text{věta o} \\ \text{separabilitě} \end{matrix} \right| = 8 a^4 b^4 \int_0^1 \rho^7 \, d\rho \int_0^{\pi/2} \cos\varphi \sin\varphi \, d\varphi =$$

$$= a^4 b^4 \left[-\frac{\cos^2(\varphi)}{2} \right]_0^{\pi/2} = \frac{1}{2} a^4 b^4$$

$$\Rightarrow \frac{\mu_2(W)}{\mu_2(O)} = \frac{1}{2} = 50\%$$

správný numerický výsledek