$$\int_0^\infty \frac{\sin^4(x)}{x^4}\,dx = \int_0^\infty \mathcal{L}^{-1}\left[\frac{1}{x^4}\right]\cdot \mathcal{L}[\sin^4(x)]\,dx$$

7 bodů

$$\mathcal{L}^{-1}\left[\frac{1}{x^4}\right] = \Theta(x)\,x^3\cdot\frac{1}{6}$$

$$\mathcal{L}[\sin^2(x)\cdot\sin^2(x)] = \mathcal{L}\left[\left(\frac{1-\cos(2x)}{2}\right)^2\right] = \frac{1}{4}\mathcal{L}\left[1-2\cos(2x)+\frac{1+\cos(4x)}{2}\right] =$$

$$= \frac{1}{8}\mathcal{L}[3-4\cos(2x)+\cos(4x)] = \frac{1}{8}\left(\frac{3}{p}-\frac{4p}{p^2+4}+\frac{p}{p^2+16}\right) =$$

$$= \frac{24}{p(p^2+4)(p^2+16)}$$

jakou větu užíváme!

$$\int_0^\infty \frac{\sin^4(x)}{x^4}\,dx = 4\int_0^\infty \frac{p^2}{(p^2+4)(p^2+16)}\,dp = 4\int_0^\infty\left(\frac{Ap+B}{p^2+4}+\frac{Cp+D}{p^2+16}\right)dp =$$

$$= \left|\begin{array}{l} (A,C)=(0,0) \\ Bp^2+16B+Dp^2+4D \overset{!}{=} p^2 \\ \left.\begin{array}{l} B+D=1 \\ 4B+D=0 \end{array}\right\} \Rightarrow (B,D)=\left(-\frac{1}{3};\frac{4}{3}\right) \end{array}\right| = \frac{16}{3}\int_0^\infty \frac{1}{p^2+16}\,dp - \frac{4}{3}\int_0^\infty \frac{1}{p^2+4}\,dp =$$

$$= \frac{1}{3}\int_0^\infty \frac{1}{\left(\frac{p}{4}\right)^2+1}\,dp - \frac{1}{3}\int_0^\infty \frac{1}{\left(\frac{p}{2}\right)^2+1}\,dp = \frac{1}{3}\left[4\,\mathrm{arctg}\frac{p}{4} - 2\,\mathrm{arctg}\frac{p}{2}\right]_0^\infty =$$

$$= \frac{1}{3}\left(4\frac{\pi}{2}-2\frac{\pi}{2}\right) = \frac{\pi}{3}$$

8 bodů

$$\mathcal{F}[e^{-cx^2}] = \sqrt{\frac{\pi}{c}}\, e^{-\frac{\xi^2}{4c}} \quad \wedge \quad \mathcal{F}[f(x-d)] = e^{id\xi}\,\mathcal{F}[f(x)]$$

$$\Rightarrow \quad \mathcal{F}[e^{-c(x-d)^2}] = \sqrt{\frac{\pi}{c}}\, e^{-\frac{\xi^2}{4c}} \cdot e^{id\xi}$$

a) Cauchyova úloha:
$$\frac{\partial u(x,t)}{\partial t} - a^2 \frac{\partial^2 u(x,t)}{\partial x^2} - b\,\frac{\partial u(x,t)}{\partial x} = f(x,t)$$
$$u(x,0) := \beta(x)$$
$$a,b>0 \wedge f(x,t) \in \mathcal{C}(t>0;\, x\in\mathbb{R}) \;\&\; u(x,t) \in \mathcal{C}^2(\mathbb{R}^{1+1})$$

b) převedení: $w(x,t) := \Theta(t)\cdot u(x,t)$

$$\frac{\partial w}{\partial t} = \delta(t) \otimes \beta(x) + \Theta(t)\cdot\frac{\partial u}{\partial t} \quad \& \quad \frac{\partial w}{\partial x} = \Theta(t)\cdot\frac{\partial u}{\partial x}$$

$$\hat{L}\, w(x,t) = \Theta(t)\cdot f(x,t) + \delta(t) \otimes \beta(x)$$

c) fundamentální řešení
$$\hat{L}\,\mathcal{E}(x,t) = \delta(x,t) = \delta(x) \otimes \delta(t) \quad / \; \mathcal{F}_x$$

$$\frac{\partial E}{\partial t} - a^2(-i\xi)^2 E - b(-i\xi) E = \delta(t)$$

$$\frac{\partial E}{\partial t} + (a^2\xi^2 + b i\xi) E = \delta(t) \quad / \; \mathcal{L}_t$$

$$p\,\mathbb{E} + (a^2\xi^2 + b i\xi)\,\mathbb{E} = 1$$

$$\mathbb{E} = \frac{1}{p + a^2\xi^2 + b i\xi} \quad \Rightarrow \quad E(\xi,t) = \Theta(t)\cdot e^{-(a^2\xi^2 + b i\xi)t}$$

$$\mathcal{E}(x,t) = \mathcal{F}_x^{-1}[\Theta(t)\, e^{-(a^2\xi^2 + 2b\xi)t}] = \Theta(t)\,\mathcal{F}_x^{-1}[e^{-a^2\xi^2 t}\cdot e^{-ib\xi t}] =$$

$$= \left| \begin{array}{l} \frac{1}{4c} = a^2 t \Rightarrow c = \frac{1}{4a^2 t} \\ d = -bt \end{array} \right| = \Theta(t)\cdot\sqrt{\frac{1}{4a^2\pi t}}\; e^{-\frac{1}{4a^2 t}(x+bt)^2}$$

$$\mathcal{E}(x,t) = \frac{\Theta(t)}{2a}\,\frac{1}{\sqrt{\pi t}}\, e^{-\frac{(x+bt)^2}{4a^2 t}}$$

– na $\mathcal{D}'_+(\mathbb{R})$ konvoluce jistě existuje mezi všemi distribucemi (s pozitivním nosičem)

– navíc definiční vztah pro konvoluci lze na $\mathcal{D}'_+(\mathbb{R})$ zjednodušit do tvaru

$$(f*g;\varphi(x)) = (f\otimes g;\eta_1(x)\eta_2(y)\varphi(x+y));$$

$\eta_j(x)\in C^\infty(\mathbb{R})$ a pro $x<K$ je $\eta_j(x)=0$

mohu volit 1 pro $\eta_1(x)$ $\forall f_k \in \mathcal{D}'_+$ (−1/2b)

$$\lim_{k\to\infty}(f_k*g;\varphi(x)) = \lim_{k\to\infty}(f_k(x)\otimes g(y);\eta_1(x)\eta_2(y)\varphi(x+y)) =$$

$$= \lim_{k\to\infty}\Big(f_k(x);\underbrace{(g(y);\eta_1(x)\eta_2(y)\varphi(x+y))}_{\in\mathcal{D}(\mathbb{R})}\Big) =$$

$$= \left|\begin{array}{l}\text{konvergence v } \mathcal{D}' \\ f_k \xrightarrow{\mathcal{D}'} f\end{array}\right| = (f(x);(g(y);\eta_1(x)\eta_2(y)\varphi(x+y))) =$$

$$= (f(x)\otimes g(y);\eta_1(x)\eta_2(y)\varphi(x+y)) \overset{\uparrow}{=}$$

(−1/2b) bez

$f_k \to f$ v $\mathcal{D}'$, $f_k \in \mathcal{D}'_+ \Rightarrow f\in\mathcal{D}'_+$

+2b

$$= (f*g;\varphi(x))$$

q.e.d.

Celkem 5 bodů

$$(f_k * g, \varphi) = (f_k \otimes g, \eta_1^k(x)\eta_2(y)\varphi(x+y))$$

$\eta_1^k, \eta_2 \in C^\infty$, $\eta_1^k \equiv 0$ na $(-\infty, -C_k)$, $C_k>0$ lib.

$\eta_2 \equiv 0$ na $(-\infty, D)$, $D>0$ lib.

a $\equiv 1$ na BÚNO okolí supp ..

BÚNO 1 η_1 pro $\forall k$

posl. rov. je pravda pouze pro $f, g \in \mathcal{D}'_+$ či finitní.

Vím že $f_k \in \mathcal{D}'_+$ & $f_k \to f$ v $\mathcal{D}'$ $\Rightarrow f\in\mathcal{D}'_+$

odpovídá 2b (def $*$, konv.)

(−3b)

2,5

$$= (\lim f_k * g, \varphi)$$

$$(f*g,\varphi) = \lim_{k\to\infty}(f(x)\otimes g(y), \eta_k(x,y)\varphi(x+y)) \overset{\checkmark}{=} \lim_{k\to\infty}\lim_{l\to\infty}(f_l(x), (g(y), \eta_k(x,y)\varphi(x+y))) \overset{\boxed{??}}{=} \lim_{l\to\infty}\lim_{k\to\infty}(f_l(x)\otimes g(y), \eta_k(x,y)\varphi(x+y))$$

1) $x^2 \notin \mathcal{S}(\mathbb{R}) \Leftarrow \sup |x^2| = +\infty$ ✓ musí být zdůvodněno

2) $x^2 \notin \mathcal{L}_1(\mathbb{R}) \Leftarrow \int_{\mathbb{R}} x^2 dx$ neexistuje, ale $\int_{\mathbb{R}} x^2 dx = +\infty$ ✓ musí být zdůvodněno

3) $x^2 \in \mathcal{S}'(\mathbb{R})$

✓ $\Leftarrow$ funkcionál $(x^2; \varphi(x))$ je lineárním a spojitým funkcionálem nad $\mathcal{S}(\mathbb{R})$ a $(x^2; \varphi(x)) = \int_{\mathbb{R}} x^2 \varphi(x) dx$ jistě existuje $\forall \varphi \in \mathcal{S}$

$\Rightarrow$ Fourierův obraz funkce x^2 existuje pouze v $\mathcal{S}'(\mathbb{R})$

$$(\mathcal{F}[x^2]; \varphi(\xi)) = (x^2; \mathcal{F}[\varphi(\xi)]) = \int_{\mathbb{R}} x^2 \mathcal{F}[\varphi(\xi)] dx = \int_{\mathbb{R}} x^2 \int_{\mathbb{R}} \varphi(\xi) e^{i\xi x} d\xi \, dx =$$ ✓

$$= \left| \begin{matrix} u = \varphi(\xi) & v' = x e^{i\xi x} \\ u' = \varphi'(\xi) & v = -i e^{i\xi x} \end{matrix} \right| = \int_{\mathbb{R}} x \left\{ \underbrace{\left[-i \varphi(\xi) e^{i\xi x} \right]_{\mathbb{R}}}_{=0 \text{ why?}} + i \int_{\mathbb{R}} \varphi'(\xi) e^{i\xi x} d\xi \right\} dx =$$ ✓

$$= i \int_{\mathbb{R}} x \int_{\mathbb{R}} \varphi'(\xi) e^{i\xi x} d\xi \, dx = \left| \begin{matrix} u = \varphi'(\xi) & v' = x e^{i\xi x} \\ u' = \varphi''(\xi) & v = -i e^{i\xi x} \end{matrix} \right| =$$

$$= i \int_{\mathbb{R}} \left\{ \underbrace{\left[-i \varphi'(\xi) e^{i\xi x} \right]_{\mathbb{R}}}_{=0} + i \int_{\mathbb{R}} \varphi''(\xi) e^{i\xi x} d\xi \right\} dx = - \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi''(\xi) e^{i\xi x} d\xi \, dx =$$ ✓

$$= - \int_{\mathbb{R}} \mathcal{F}[\varphi''(\xi)] dx = - (1; \mathcal{F}[\varphi''(\xi)]) = - (\mathcal{F}[\delta]; \mathcal{F}[\varphi''(\xi)]) =$$ ✓

$$= - (\mathcal{F}\mathcal{F}[\delta]; \varphi''(\xi)) = - (\sqrt{2\pi}\, \delta(-\xi); \varphi''(\xi)) = -\sqrt{2\pi} (\delta(\xi); \varphi''(\xi)) =$$ ✓

$$= -\sqrt{2\pi} (\delta''(\xi); \varphi(\xi))$$

$$\Rightarrow \quad \mathcal{F}[x^2] = -\sqrt{2\pi}\, \delta''(\xi)$$ ✓

$$\mathcal{L}[\theta(x)] = \frac{1}{p} \qquad \mathcal{L}[\theta(x)\cdot x^n] = \left| \mathcal{L}[x^n f(x)] = (-1)^n \frac{d^n F}{dp^n} \right| = (-1)^n \cdot (-1)^n \cdot \frac{n!}{p^{n+1}}$$

$$\mathcal{L}[\theta(x)\cdot x^n] = \frac{n!}{p^{n+1}}$$

$$\mathcal{L}\left[\theta(x) \frac{x^{n-1}}{(n-1)!}\right] = \frac{1}{(n-1)!} \frac{(n-1)!}{p^{n+1-1}} = \frac{1}{p^n} \checkmark$$

$$\mathcal{L}\left[\theta(x) \frac{x^{n-1}}{(n-1)!} e^{Dx}\right] = \left| \mathcal{L}[e^{at} f(x)] = F(p-a) \right| = \frac{1}{(p \pm D)^n} \checkmark$$

$$\mathcal{L}\left[\theta(x) \frac{x^{n-1}}{(n-1)!} e^{Dx} * \theta(x) \frac{x^{m-1}}{(m-1)!} e^{Dx}\right] \overset{\checkmark}{=} \frac{1}{(p+D)^n} \cdot \frac{1}{(p+D)^m} = \frac{1}{(p+D)^{n+m}} \checkmark$$

$$f_n(x) * f_m(x) = \theta(x) \frac{x^{m+n-1}}{(m+n-1)!} e^{Dx} = f_{m+n}(x)$$

✓

✓ za konec: $f_n(x) * f_m(x) = f_{m+n}(x)$ ✓

celkem 6 bodů

7b

$$\left(\frac{d}{dx}\left(\Theta_\mu(x)\otimes h(t)\right); \varphi(x,t)\right) = -\left(\Theta_\mu(x)\otimes h(t); \frac{\partial\varphi}{\partial x}(x,t)\right) =$$

$$= -\left(\Theta_\mu(x); \left(h(t); \frac{\partial\varphi}{\partial t}(x,t)\right)\right) = \left| \begin{array}{c}\text{podle důležité věty}\\ \text{z přednášky}\end{array} \right| =$$

$$= -\left(\Theta_\mu(x); \frac{d}{dx}\left(h(t), \varphi(x,t)\right)\right) = (-1)^2\left(\Theta_\mu'(x); \left(h(t); \varphi(x,t)\right)\right) =$$

+ vysvětlení

$$= \left(\Theta_\mu' \otimes h(t); \varphi(x,t)\right) = \left| \begin{array}{c}\text{odvození}\\ \text{níže}\end{array} \right| = \left(\delta_\mu(x)\otimes h(t); \varphi(x,t)\right)$$

$$\left(\Theta_\mu'; \varphi(x)\right) = -\left(\Theta_\mu; \varphi'(x)\right) = -\int_\mu^{+\infty} \varphi'(x)\,dx = -\left[\varphi(x)\right]_\mu^{+\infty} =$$

$$= 0 + \varphi(\mu) = \varphi(\mu) = \left(\delta_\mu; \varphi(x)\right)$$

$$\left(\frac{d}{dx}\left(\Theta(x)\otimes h(t)\right); \varphi(x,t)\right) = \left(\delta(x)\otimes h(t); \omega_\varepsilon(x,t)\right) =$$

$$= \left(\delta(x)\otimes t^3; \omega_\varepsilon(x,t)\right) = \left(\delta(x); \left(t^3; \omega_\varepsilon(x,t)\right)\right) =$$

$$= \left(t^3; \omega_\varepsilon(0,t)\right) = \int_{-\varepsilon}^{\varepsilon} t^3 \cdot C_\varepsilon \cdot e^{-\frac{\varepsilon^2}{\varepsilon^2 - \|(0,t)\|^2}}\,dt = C_\varepsilon \int_{-\varepsilon}^{\varepsilon} t^3 \cdot e^{-\frac{\varepsilon^2}{\varepsilon^2 - t^2}}\,dt =$$

$$= \left| \begin{array}{l}\text{lichý integrand \&}\\ \text{integrační množina symetrická okolo } x=0\end{array} \right| = 0$$

$$g(x|a,\mu) = \sqrt{\frac{a}{\pi}}\, e^{-a(x-\mu)^2}$$

$$\mathcal{F}[g(x|a,\mu) * g(x|a,\mu)] \overset{\text{metoda}}{=} \mathcal{F}^2[g(x|a,\mu)] = \underbrace{\left(\sqrt{\frac{a}{\pi}}\, e^{i\mu\xi} \sqrt{\frac{\pi}{a}}\, e^{-\frac{\xi^2}{4a}}\right)^2}_{\text{obraz} \checkmark} =$$

$$= e^{2i\mu\xi}\, e^{-\frac{\xi^2}{2a}} \checkmark$$

$$\mathcal{F}[e^{-\frac{x^2}{2a}}] = \sqrt{\pi \cdot 2a} \cdot e^{-2a\xi^2 \cdot \frac{1}{4}} = \sqrt{2\pi a}\, e^{-a\frac{\xi^2}{2}}$$

$$\mathcal{F}[e^{2i\mu x} \cdot e^{-\frac{x^2}{2a}}] = \sqrt{2\pi a}\, e^{-\frac{(\xi+2\mu)^2}{2} a}$$

$$\mathcal{F}\mathcal{F}[f(x)] = 2\pi f(-x) \Rightarrow \mathcal{F}[f(x)] = 2\pi \mathcal{F}^{-1}[f(-x)]$$

$$\Rightarrow \mathcal{F}^{-1}[f(x)] = \frac{1}{2\pi} \mathcal{F}[f(-x)]$$

$\Rightarrow$ metoda hledání vzoru ✓

$$\mathcal{F}^{-1}[e^{2i\mu\xi}\, e^{-\frac{\xi^2}{2a}}] = \frac{1}{2\pi} \mathcal{F}[e^{-2i\mu\xi}\, e^{-\frac{\xi^2}{2a}}] =$$

$$= \frac{1}{2\pi}\sqrt{2\pi a}\, e^{-\frac{(x-2\mu)^2}{2} a} = \sqrt{\frac{a}{2\pi}}\, e^{-\frac{a}{2}(x-2\mu)^2} \checkmark$$

<u>Závěr:</u>

$$g(x|a,\mu) * g(x|a,\mu) = g(x|\tfrac{a}{2}; 2\mu) \checkmark$$

Celkem 6 bodů

$$\left(\frac{d}{dx}\left(\theta_\mu(x)\otimes\beta(t)\right);\varphi(x,t)\right)=-\left(\theta_\mu(x)\otimes\beta(t);\frac{\partial\varphi}{\partial x}(x,t)\right)=$$

$$=-\left(\theta_\mu(x);\left(\beta(t);\frac{\partial\varphi}{\partial x}(x,t)\right)\right)=\left|\begin{array}{c}\text{podle důležité věty}\\ \text{z přednášek}\end{array}\right|=$$

$$=-\left(\theta_\mu(x);\frac{d}{dx}\left(\beta(t);\varphi(x,t)\right)\right)=(-1)^2\left(\theta_\mu'(x);\left(\beta(t);\varphi(x,t)\right)\right)=$$

+ vysvětlení

$$=\left(\theta_\mu'\otimes\beta(t);\varphi(x,t)\right)=\left|\begin{array}{c}\text{odvození}\\ \text{níže}\end{array}\right|=\left(\delta_\mu(x)\otimes\beta(t);\varphi(x,t)\right)$$

$$\left(\theta_\mu';\varphi(x)\right)=-\left(\theta_\mu;\varphi'(x)\right)=-\int_\mu^{+\infty}\varphi'(x)\,dx=-\left[\varphi(x)\right]_\mu^{+\infty}=$$

$$=0+\varphi(\mu)=\varphi(\mu)=\left(\delta_\mu;\varphi(x)\right)$$

Cimrmanova čepička centrovaná do (μ,μ):

$$\omega_\varepsilon(x,y)=\begin{cases}C_\varepsilon\, e^{-\frac{\varepsilon^2}{\varepsilon^2-\|(x-\mu;y-\mu)\|^2}} & \|(x-\mu;y-\mu)\|<\varepsilon\\ 0 & \|(x-\mu;y-\mu)\|\geq 0\end{cases}$$

$$\left(\delta_\mu(x)\otimes\beta(t);\omega_\varepsilon(x,t)\right)=\left(\delta_\mu(x)\otimes(t-\mu)^5;\omega_\varepsilon(x,t)\right)=$$

$$=\left(\delta_\mu(x);\left((t-\mu)^5;\omega_\varepsilon(x,t)\right)\right)=\left((t-\mu)^5;\omega_\varepsilon(\mu;t)\right)=$$

$$=\int_{\mu-\varepsilon}^{\mu+\varepsilon}(t-\mu)^5 e^{-\frac{\varepsilon^2}{\varepsilon^2-(t-\mu)^2}}\,dt=\left|\begin{array}{c}y=t-\mu\\ dy=dt\end{array}\right|=\int_{-\varepsilon}^{\varepsilon}y^5 e^{-\frac{\varepsilon^2}{\varepsilon^2-y^2}}\,dy=$$

$$=\left|\begin{array}{l}\text{lichý integrand \&}\\ \text{integrační množina symetrická okolo } y=0\end{array}\right|=0$$

$$\int_0^\infty \frac{e^{-\alpha x}-e^{-\beta x}}{x}\sin(\gamma x)\,dx \qquad \alpha,\beta,\gamma>0$$

$$f(x)=(e^{-\alpha x}-e^{-\beta x})\sin(\gamma x) \quad \& \quad G(p)=\frac{1}{p}$$

$$\mathcal{L}[\sin(\gamma x)]=\frac{\gamma}{\gamma^2+p^2} \Rightarrow \mathcal{L}[e^{-\alpha x}\sin(\gamma x)-e^{-\beta x}\sin(\gamma x)]=\underbrace{\frac{\gamma}{\gamma^2+(p+\alpha)^2}-\frac{\gamma}{\gamma^2+(p+\beta)^2}}$$

$$\mathcal{L}^{-1}[G(p)]=\theta(x)$$

metoda

$$\int_0^\infty \frac{e^{-\alpha x}-e^{-\beta x}}{x}\sin(\gamma x)\,dx=\int_0^\infty \frac{\gamma}{\gamma^2+(p+\alpha)^2}\,dp-\int_0^\infty \frac{\gamma}{\gamma^2+(p+\beta)^2}\,dp=$$

$$=\frac{1}{\gamma}\int_0^\infty \frac{1}{1+\left(\frac{p+\alpha}{\gamma}\right)^2}\,dp-\frac{1}{\gamma}\int_0^\infty \frac{1}{1+\left(\frac{p+\beta}{\gamma}\right)^2}\,dp=$$

$$=\left[\operatorname{arctg}\frac{p+\alpha}{\gamma}-\operatorname{arctg}\frac{p+\beta}{\gamma}\right]_0^\infty=\frac{\pi}{2}-\operatorname{arctg}\frac{\alpha}{\gamma}-\frac{\pi}{2}+\operatorname{arctg}\frac{\beta}{\gamma}=$$

$$=\operatorname{arctg}\frac{\beta}{\gamma}-\operatorname{arctg}\frac{\alpha}{\gamma}$$

chyba $\frac{1}{\gamma}\int\frac{1}{1+\left(\frac{p+\alpha}{\gamma}\right)^2}\,dp=\frac{1}{\gamma}\operatorname{arctg}\frac{p+\alpha}{\gamma}$

se netoleruje !!