do $\mathcal{S}$ patří např. funkce $\varphi(x) = e^{-x^2}$ & $\mathcal{D}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$ 4b

$\Rightarrow$ funkce v $\mathcal{S}$ nemusejí mít omezený nosič & v tom může být problém

$\mathcal{D} \subset \mathcal{S} \Rightarrow \mathcal{S}' \subset \mathcal{D}'$

např. $e^{x^2} \in \mathcal{D}'(\mathbb{R})$, neboť $\int_{\mathbb{R}} e^{x^2} \varphi(x)\, dx$ existuje $\forall \varphi(x) \in \mathcal{D}$,

ale $\int_{\mathbb{R}} e^{x^2} \varphi(x)\, dx$ neexistuje např. právě pro $\varphi(x) = e^{-x^2}$

$\Rightarrow e^{x^2} \in \mathcal{S}' \setminus \mathcal{D}'$

$$\lim_{\omega\to\infty} \frac{\operatorname{arctg}(a\omega x)}{x(1+\omega^2x^2)} = ? \quad \text{v} \quad \mathcal{D}'(\mathbb{R})$$

$$\lim_{\omega\to\infty}\left(\frac{\operatorname{arctg}(a\omega x)}{x(1+\omega^2x^2)};\ \varphi(x)\right) = \left|\begin{array}{l}\text{Je funkce } \tilde{f} \text{ regulární?} \\ \lim\limits_{x\to 0}\dfrac{\operatorname{arctg}(a\omega x)}{x} = a\omega\end{array} \quad \begin{array}{l} f(x) \text{ je tedy ekvivalentní s} \\ \text{jistou spojitou funkcí} \Rightarrow \\ \Rightarrow f(x) \text{ je lokálně integrabilní}\end{array}\right| =$$

$$= \lim_{\omega\to\infty}\int_{\mathbb{R}} \frac{\operatorname{arctg}(a\omega x)}{x(1+\omega^2x^2)}\,\varphi(x)\,dx = \left|\begin{array}{l} y=\omega x \\ dx = \frac{1}{\omega}dy\end{array}\right| = \lim_{\omega\to\infty}\int_{\mathbb{R}}\frac{\operatorname{arctg}(ay)}{\frac{y}{\omega}(1+y^2)}\,\varphi\!\left(\frac{y}{\omega}\right)\frac{1}{\omega}\,dy =$$

$$= \lim_{\omega\to\infty}\int_{\mathbb{R}}\frac{\operatorname{arctg}(ay)}{y(1+y^2)}\,\varphi\!\left(\frac{y}{\omega}\right)dy =$$

$$= \left|\ \left|\frac{\operatorname{arctg}(ay)}{y(1+y^2)}\,\varphi\!\left(\frac{y}{\omega}\right)\right| \leq \frac{K}{1+y^2}\left|\frac{\operatorname{arctg}(ay)}{y}\right| \leq \frac{K\cdot a}{1+y^2} \in \mathcal{L}(\mathbb{R})\ \right| =$$

- pro $\omega\to\infty$ nemusí mít $\varphi\left(\frac{y}{\omega}\right)$ omezený nosič
- funkce $\frac{\operatorname{arctg}(ay)}{y}$ má extrém v bodě $y=0$ rovný číslu a (jediný extrém!)
- $\lim\limits_{y\to\pm\infty}\frac{\operatorname{arctg}(ay)}{y} = 0$

$$= \int_{\mathbb{R}}\frac{\operatorname{arctg}(ay)}{y(1+y^2)}\,\varphi(0)\,dy = \varphi(0)\cdot\overbrace{\int_{\mathbb{R}}\frac{\operatorname{arctg}(ay)}{y(1+y^2)}\,dy}^{G(a)}$$

$$\frac{dG}{da} = \left|\begin{array}{l} G(0)=0 \\ \text{integrand } \lambda\text{-měřitelný} \Leftarrow \text{ekvivalentní s jistou spojitou funkcí} \\ \left|\dfrac{\partial g}{\partial a}\right| = \left|\dfrac{1}{(1+a^2y^2)(1+y^2)}\right| \leq \dfrac{1}{1+y^2} \in \mathcal{L}(\mathbb{R}) \ \dots \ \text{majoranta nezávislá na parametru}\end{array}\right| =$$

$$= \int_{\mathbb{R}}\frac{1}{(1+a^2y)(1+y^2)}\,dy = \int_{\mathbb{R}}\frac{Ay+B}{1+a^2y^2}\,dy + \int_{\mathbb{R}}\frac{Cy+D}{1+y^2}\,dy = \left|\begin{array}{l} Ay+B+Ay^3+By^2+Cy+D+Ca^2y^3+ \\ +Da^2y^2 \overset{!}{=} 1 \\ (A,C)=(0,0)\ \&\ B+D=1\ \& \\ \&\ B+Da^2=0\end{array}\right| =$$

$$= \left|\begin{array}{l} B=\dfrac{a^2}{a^2-1} \\ D=-\dfrac{1}{a^2-1}\end{array}\right| = \frac{a^2}{a^2-1}\int_{\mathbb{R}}\frac{1}{1+a^2y}\,dy - \frac{1}{a^2-1}\int_{\mathbb{R}}\frac{1}{1+y^2}\,dy = \frac{a}{a^2-1}\left[\operatorname{arctg}(ay)\right]_{-\infty}^{\infty} - \frac{1}{a^2-1}\left[\operatorname{arctg} y\right]_{-\infty}^{\infty} =$$

$$= \pi\left(\frac{a}{a^2-1} - \frac{1}{a^2-1}\right) = \pi\,\frac{1}{a+1} \Rightarrow G(a) = \pi\ln(a+1) + C \ \ \&\ \ C=0$$

$$\Rightarrow \lim_{\omega\to\infty}\tilde{f}_a = \pi\ln(a+1)\,\tilde{\delta}$$

$$\varphi(x) = \mu \int_0^x \frac{x^2 \varphi(y)}{y}\, dy + x^2$$

$\longrightarrow$ pochopení metody ✓

$$\varphi_0(x) = x^2$$

$$\varphi_1(x) = \mu \int_0^x \frac{x^2}{y} y^2\, dy + x^2 = \mu x^2 \left[\frac{y^2}{2}\right]_0^x + x^2 = \frac{1}{2}\mu x^4 + x^2$$

$$\varphi_2(x) = \mu \int_0^x \frac{x^2}{y} \frac{1}{2}\mu y^4\, dy + \frac{1}{2}\mu \cdot x^4 + x^2 = \frac{1}{2}\cdot\frac{1}{4}\cdot\mu^2 x^6 + \frac{1}{2}\mu x^4 + x^2$$

$$\varphi_3(x) = \mu \int_0^x \frac{x^2}{y} \frac{1}{2}\frac{1}{4}\mu^2 y^6\, dy + \frac{1}{2}\cdot\frac{1}{4}\cdot\mu^2 x^6 + \frac{1}{2}\mu x^4 + x^2 =$$

$$= \frac{1}{2}\cdot\frac{1}{4}\cdot\mu^3 \frac{1}{6} x^8 + \frac{1}{2}\mu x^4 + \frac{1}{2}\cdot\frac{1}{4}\mu^2 x^6 + x^2$$

Podezření: $\displaystyle \varphi_n(x) = \sum_{k=0}^{n} \frac{\mu^k}{(2k)!!} x^{2+2k}$ ✓✓

Důkaz:

$$\varphi_{n+1}(x) = \mu \int_0^x \frac{x^2}{y} \varphi_n(y)\, dy + x^2 = \mu \int_0^x \frac{x^2}{y} \sum_{k=0}^{n} \frac{\mu^k}{(2k)!!} y^{2+2k}\, dy + x^2 =$$

$$= \sum_{k=0}^{n} x^2 \mu^{k+1} \frac{1}{(2k)!!} \int_0^x y^{1+2k}\, dy + x^2 = x^2 \sum_{k=0}^{n} \frac{\mu^{k+1}}{(2k)!!} \left[\frac{y^{2+2k}}{2+2k}\right]_0^x + x^2 =$$

$$= \sum_{k=0}^{n} \frac{\mu^{k+1}}{(2k+2)!!} x^{4+2k} + x^2 = \sum_{l=1}^{n+1} \frac{\mu^l}{(2l)!!} x^{2(l+1)} + x^2 = \sum_{l=0}^{n+1} \frac{\mu^l}{(2l)!!} x^{2(l+1)}$$

✓✓

Řešení:

$$\varphi(x) = \lim_{n\to\infty} \varphi_n(x) = \sum_{l=0}^{\infty} \frac{\mu^l}{(2l)!!} x^{2(l+1)} = x^2 \sum_{l=0}^{\infty} \frac{\mu^l \cdot x^{2l}}{2^l \cdot l!} =$$

$$= x^2 \sum_{l=0}^{\infty} \left(\frac{\mu x^2}{2}\right)^l \cdot \frac{1}{l!} = x^2 e^{\frac{\mu x^2}{2}}$$

✓✓

8b

$$x^2 \frac{\partial^2 u}{\partial x^2} + 4y^2 \frac{\partial^2 u}{\partial y^2} + 4xy \frac{\partial^2 u}{\partial x \partial y} - x \frac{\partial u}{\partial x} + u(x,y) = 0$$

$d(x,y) = 16x^2y^2 - 16x^2y^2 = 0 \quad \Rightarrow \quad$ všude parabolická

$$\lambda(x,y) = -\frac{4xy}{2x^2} = -2\frac{y}{x} \quad \Rightarrow \quad y' = 2\frac{y}{x} \quad \Rightarrow \quad \ln|y| = \ln Cx^2$$

$$\xi = \frac{x^2}{y} \qquad \det\left(\frac{D(\xi,\eta)}{D(x,y)}\right) = \begin{vmatrix} \frac{2x}{y} & -\frac{x^2}{y^2} \\ 1 & 0 \end{vmatrix} = \frac{2x}{y}$$

$$\eta = x$$

$$M_{reg} = \{(x,y) \in \mathbb{R}^2 : x \neq 0 \wedge y \neq 0\}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} 2\frac{x}{y} + \frac{\partial u}{\partial \eta} \quad \Big| -x \qquad \frac{\partial u}{\partial y} = -\frac{x^2}{y^2}\frac{\partial u}{\partial \xi} \quad \Big| 0$$

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{x^2}{y^2}\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + 4\frac{x}{y}\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{2}{y}\frac{\partial u}{\partial \xi} \quad \Big| x^2$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^4}{y^4}\frac{\partial^2 u}{\partial \xi^2} + 2\frac{x^2}{y^3}\frac{\partial u}{\partial \xi} \quad \Big| 4y^2$$

$$\frac{\partial^2 u}{\partial x \partial y} = -2\frac{x^3}{y^3}\frac{\partial^2 u}{\partial \xi^2} - \frac{x^2}{y^2}\frac{\partial^2 u}{\partial \xi \partial \eta} - 2\frac{x}{y^2}\frac{\partial u}{\partial \xi} \quad \Big| 4xy$$

$$\frac{\partial^2 u}{\partial \xi^2}\left[4\frac{x^4}{y^2} + 4\frac{x^4}{y^2} - 8\frac{x^4}{y^2}\right] + x^2\frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 u}{\partial \xi \partial \eta}\left[4\frac{x^3}{y} - 4\frac{x^3}{y}\right] + \frac{\partial u}{\partial \xi}\left[2\frac{x^2}{y} + 8\frac{x^2}{y} - 8\frac{x^2}{y} - 2\frac{x^2}{y}\right]$$

$$-x\frac{\partial u}{\partial \eta} + u(\xi,\eta) = 0 \qquad x^2\frac{\partial^2 u}{\partial \eta^2} - x\frac{\partial u}{\partial \eta} + u = 0$$

$$\eta^2\frac{\partial^2 u}{\partial \eta^2} - \eta\frac{\partial u}{\partial \eta} + u = 0$$

- $\eta > 0$

$$\eta = e^t \Rightarrow t = \ln\eta \quad \Rightarrow \quad \frac{du}{d\eta} = \frac{du}{dt}\frac{1}{\eta} \quad \Rightarrow \quad \frac{d^2u}{d\eta^2} = -\frac{1}{\eta^2}\dot{u} + \ddot{u}\frac{1}{\eta^2}$$

$$\ddot{u} - \dot{u} - \dot{u} + u = 0 \quad \Leftrightarrow \quad \ddot{u} - 2\dot{u} + u = 0$$

$$\lambda^2 - 2\lambda + 1 = (\lambda-1)^2$$

$$F_S = \{e^t; te^t\} \quad \Rightarrow \quad F_S' = \{\eta; \eta \cdot \ln(\eta)\}$$

- $\eta < 0$ analogicky $\qquad u(\xi,\eta) = C(\xi)\cdot\eta + D(\xi)\cdot\eta\cdot\ln|\eta|$

Závěr:

$$u(x,y) = C\left(\frac{x^2}{y}\right)\cdot x + D\left(\frac{x^2}{y}\right)\cdot x \cdot \ln|x| \quad \text{na } M_{reg}$$

$$\hat{K} = \int_0^\infty (xy + y^2) e^{-ty^2} \cdot dy$$

$$\hat{K}\varphi(x) = \int_0^\infty (xy+y^2) e^{-ty^2} \varphi(y)\, dy = \lambda \varphi(x)$$

separace

$$\underbrace{x \int_0^\infty y e^{-ty^2} \varphi(y)\, dy}_{A} + \underbrace{\int_0^\infty y^2 e^{-ty^2} \varphi(y)\, dy}_{B} = \lambda \varphi(x)$$

$$\lambda \varphi(x) = Ax + B \quad \Big/ \int_0^\infty x e^{-tx^2} \cdot dx \quad \& \quad \int_0^\infty x^2 e^{-tx^2} \cdot dx$$

$$\lambda A = A \int_0^\infty x^2 e^{-tx^2} dx + B \int_0^\infty x e^{-tx^2} dx$$

$$\lambda B = A \int_0^\infty x^3 e^{-tx^2} dx + B \int_0^\infty x^2 e^{-tx^2} dx$$

$$\int_0^\infty x e^{-tx^2} dx = \left| \begin{matrix} u = x^2 \\ du = 2x\, dx \end{matrix} \right| = \frac{1}{2} \int_0^\infty e^{-tu} du = \frac{1}{2t}$$

$$\int_0^\infty e^{-tx^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{t}} \Rightarrow \int_0^\infty x^2 e^{-tx^2} dx = \frac{1}{4}\sqrt{\frac{\pi}{t^3}}$$

$$\int_0^\infty x e^{-tx^2} dx = \frac{1}{2t} \Rightarrow \int_0^\infty x^3 e^{-tx^2} dx = \frac{1}{2t^2}$$

$$\left.\begin{aligned} \lambda A &= \frac{A}{4}\frac{\sqrt{\pi}}{t^{3/2}} + \frac{B}{2t} \\ \lambda B &= \frac{A}{2t^2} + B\,\frac{1}{4}\frac{\sqrt{\pi}}{t^{3/2}} \end{aligned}\right\} \Rightarrow \begin{pmatrix} \frac{\sqrt{\pi}}{4t^{3/2}} - \lambda & \frac{1}{2t} \\ \frac{1}{2t^2} & \frac{\sqrt{\pi}}{4t^{3/2}} - \lambda \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} \frac{\sqrt{\pi}}{4t^{3/2}} - \lambda & \frac{1}{2t} \\ \frac{1}{2t^2} & \frac{\sqrt{\pi}}{4t^{3/2}} - \lambda \end{vmatrix} = \left(\frac{\sqrt{\pi}}{4t^{3/2}} - \lambda\right)^2 - \frac{1}{4t^3} \stackrel{!}{=} 0$$

$$\left(\frac{\sqrt{\pi}}{4} - \lambda t^{3/2}\right)^2 = \frac{1}{4}$$

$$\left|\frac{\sqrt{\pi}}{4} - \lambda t^{3/2}\right| = \frac{1}{2} \Rightarrow |\sqrt{\pi} - 4\lambda t^{3/2}| = 2$$

$$\lambda_1 = \frac{-2+\sqrt{\pi}}{4t^{3/2}} \quad \& \quad \lambda_2 = \frac{2+\sqrt{\pi}}{4t^{3/2}}$$

a) existence

$$\int_0^\infty g(x)\varphi(x)\,dx = \int_0^R g(x)\varphi(x)\,dx = \int_{\langle 0,R\rangle = K} g(x)\varphi(x)\,dx$$

$$|g(x)\varphi(x)| \leq K\cdot \underbrace{|g(x)| \in \mathcal{L}(\langle 0,R\rangle)}_{\text{z definice, resp. z věty o lokální integrabilitě}}$$

$\Rightarrow$ ze srovnávacího kritéria platí, že $g(x)\varphi(x) \in \mathcal{L}(0;+\infty)$

b) linearita

c) spojitost

$$\varphi_k(x) \rightrightarrows \varphi(x) \overset{?}{\Rightarrow} \lim_{k\to\infty} (\tilde{g};\varphi_k(x)) = (\tilde{g};\varphi(x))$$

•) stačí zkoumat pro $\varphi(x)=0$

$$\lim_{k\to\infty}(\tilde{g};\varphi_k(x)) = \lim_{k\to\infty}\int_0^\infty g(x)\varphi_k(x)\,dx = \left|\begin{array}{l}\text{z definice superstejn.}\\ \text{konvergence, zejména ze}\\ \text{stejnoměrné omezenosti}\end{array}\right| =$$

$$= \lim_{k\to\infty}\int_0^R g(x)\varphi_k(x)\,dx = \Bigg|$$ je třeba hledat majorantu nezávislou na k

$$|g(x)\varphi_k(x)| \leq K\cdot|g(x)|$$

$$\varphi_k(x) \rightrightarrows \varphi(x) \Rightarrow \forall\varepsilon>0\ \exists n_0:\ n>n_0 \wedge x\in\langle 0,R\rangle\ |\varphi_k(x)|<\varepsilon$$

$$= \int_0^R g(x)\lim_{k\to\infty}\varphi_k(x)\,dx = \int_0^R 0\,dx = 0$$ $\Rightarrow$ dokázána spojitost

d)

$$(\tilde{g}',\varphi(x)) = -(\tilde{g};\varphi'(x)) = -\int_0^\infty g(x)\varphi'(x)\,dx = \left|\begin{array}{ll} u=g & v'=\varphi' \\ u'=g' & v=\varphi\end{array}\right| =$$

$$= -[g(x)\varphi(x)]_0^\infty + \int_0^\infty g'(x)\varphi(x)\,dx = -g(0_+)\varphi(0) + \int_0^\infty g'(x)\varphi(x)\,dx =$$

$$= (\tilde{g}^\# - g(0_+)\tilde{\delta};\varphi(x))$$

$$\tilde{g}' = \tilde{g}^\# - g(0_+)\tilde{\delta}$$ $\tilde{g}^\#$ je distribuce generovaná funkcí $g'(x)$ na $(0;+\infty)$

[illegible] %.

$G \subset R$

$$\|f\|_G = \max_{x\in\overline{G}} |f(x)|$$

Zvolme libovolnou cauchyovskou posloupnost $(f_n(x))_{n=1}^{\infty}$. Platí tedy

$$\forall \varepsilon>0\ \exists n_0 \in N: \quad m,n > n_0 \Rightarrow \underbrace{\|f_n(x) - f_m(x)\|_G < \varepsilon}$$

$$\max_{x\in\overline{G}} |f_n(x) - f_m(x)| < \varepsilon$$

$\Rightarrow \forall x \in \overline{G}: \ |f_n(x) - f_m(x)| < \varepsilon \Rightarrow$ to ale znamená, že v každém

bodě $x \in \overline{G}$ je posloupnost (číselná posloupnost) $(f_n(x))_{n=1}^{\infty}$ cauchyovská $\Rightarrow$

$\Rightarrow f_n(x)$ je konvergentní jako číselná posloupnost $\Rightarrow \forall x \in \overline{G}: \lim\limits_{n\to\infty} f_n(x) =: f(x)$

existence

Ukažme, že tato bodová limita $f(x)$ je zároveň limitou ve smyslu konvergence v metrickém prostoru $C(\overline{G})$ s metrikou $\|\cdot\|_G$.

$$\|f_n(x) - f_m(x)\|_G = \max_{x\in\overline{G}} |f_n(x) - f_m(x)| < \frac{\varepsilon}{2} \quad \Big/ \begin{matrix} n \text{ fixujeme} \\ m \to +\infty \end{matrix}$$

$$\max_{x\in\overline{G}} |f_n(x) - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon$$

$$\Downarrow$$

$$\|f_n(x) - f(x)\|_G < \varepsilon$$

$$\hat{K} = \int_0^\infty y(x+y)\, e^{-ty} \bullet\, dy \qquad t>0$$

$$\hat{K}\varphi(x) = \int_0^\infty y(x+y)\, e^{-ty} \varphi(y) = \lambda \varphi(x)$$

separace

$$x \cdot \underbrace{\int_0^\infty y\, e^{-ty} \varphi(y)\, dy}_{A} + \underbrace{\int_0^\infty y^2 e^{-ty} \varphi(y)\, dy}_{B} = \lambda \varphi(x)$$

$$\lambda \varphi(x) = Ax + B \quad \Big/ \int_0^\infty x\, e^{-tx} \bullet\, dx \quad \Big/ \int_0^\infty x^2 e^{-tx} \bullet\, dx$$

$$\lambda A = A \int_0^\infty x^2 e^{-tx}\, dx + B \int_0^\infty x\, e^{-tx}\, dx$$

$$\lambda B = A \int_0^\infty x^3 e^{-tx}\, dx + B \int_0^\infty x^2 e^{-tx}\, dx$$

$$\int_0^\infty e^{-ax}\, dx = \frac{1}{a} \Rightarrow \int_0^\infty x\, e^{-ax}\, dx = \frac{1}{a^2} \Rightarrow \int_0^\infty x^2 e^{-ax}\, dx = \frac{2}{a^3} \Rightarrow$$

$$\Rightarrow \int_0^\infty x^3 e^{-ax}\, dx = \frac{6}{a^4} \Rightarrow \int_0^\infty x^4 e^{-ax}\, dx = \frac{24}{a^5}$$

Dosazení!

$$\lambda A = A \frac{2}{t^3} + \frac{B}{t^2}$$

$$\lambda B = A \cdot \frac{6}{t^4} + B \frac{2}{t^3} \Rightarrow \begin{pmatrix} \frac{2}{t^3} - \lambda & \frac{1}{t^2} \\ \frac{6}{t^4} & \frac{2}{t^3} - \lambda \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

$$\begin{vmatrix} \frac{2}{t^3} - \lambda & \frac{1}{t^2} \\ \frac{6}{t^4} & \frac{2}{t^3} - \lambda \end{vmatrix} = \left(\frac{2}{t^3} - \lambda\right)^2 - \frac{6}{t^6} \overset{!}{=} 0$$

$$(2 - t^3 \cdot \lambda)^2 = 6$$

$$|2 - \lambda \cdot t^3| = \sqrt{6}$$

$$\lambda_1 = \frac{2+\sqrt{6}}{t^3} \;\&\; \lambda_2 = \frac{2-\sqrt{6}}{t^3}$$

a $\lambda_3 = 0$ nekonečně násobná vl. h.

+1b

Je funkce $\tilde{f}_\omega$ regulární?

$$\lim_{x\to 0_+} \frac{\cos^2(c\omega x)-\cos^2(\omega x)}{x} \overset{L'H}{=} \lim_{x\to 0_+}\left[2\cos(c\omega x)\cdot c\cdot\omega\cdot\sin(c\omega x)-2\cos(\omega x)\sin(\omega x)\omega\right]=0$$

$$\lim_{x\to 0_-} \Theta(x)e^{-x\omega}\frac{\cos^2(c\omega x)-\cos^2(\omega x)}{x}=0 \quad\Rightarrow\quad f_\omega(x) \text{ je spojitá v } \mathbb{R} \Rightarrow$$

✓ bonus $\Rightarrow$ lokálně integrabilní $\Rightarrow \tilde{f}_\omega \in \mathcal{D}'_{reg}$

$$\lim_{\omega\to\infty}(\tilde{f}_\omega;\varphi(x)) = \int_{\mathbb{R}}\Theta(x)e^{-x\omega}\frac{\cos^2(c\omega x)-\cos^2(\omega x)}{x}\varphi(x)\,dx = \left|\begin{array}{l}\omega x=y\\ dx=\frac{1}{\omega}dy\end{array}\right. \;\&\; x<0\Rightarrow y<0\Rightarrow\Theta(y)=0$$

✓ ✓

$$=\lim_{\omega\to\infty}\int_0^\infty e^{-y}\frac{\cos^2(cy)-\cos^2(y)}{y\frac{1}{\omega}}\varphi\left(\frac{y}{\omega}\right)\frac{dy}{\omega} = \lim_{\omega\to\infty}\int_0^\infty e^{-y}\underbrace{\frac{\cos^2(cy)-\cos^2(y)}{y}}_{h(y)}\varphi\left(\frac{y}{\omega}\right)dy =$$

✓

$$= \left|\; \left|e^{-y}\frac{\cos^2(cy)-\cos^2(y)}{y}\varphi\left(\frac{y}{\omega}\right)\right| \leq K\cdot e^{-y}\left|\frac{\cos^2(cy)-\cos^2(y)}{y}\right| \leq K\cdot L\cdot e^{-y}\in\mathcal{L}(\mathbb{R}^+)\right. =$$

$$h'(y)\cdot y^2 = [-2\cos(cy)\sin(cy)c+2\cos(y)\sin(y)]y-[\cos^2(cy)-\cos^2(y)] \overset{!}{=} 0$$

derivace je složitá, ale funkce je spojitá a její amplituda klesá k nule $\Rightarrow$ $\int_0^1$ spoj. fce $<+\infty$; $+\int_1^\infty |\;| \leq Ke^{-y}\in\mathcal{L}(\mathbb{R}^+)$

$\Rightarrow$ $|h(y)|$ je omezená!!

(✓✓ zdůvodnění)

$$= \varphi(0)\cdot\underbrace{\int_0^\infty \overbrace{e^{-y}\frac{\cos^2(cy)-\cos^2(y)}{y}}^{g(y|c)}dy}_{G(c)} = \varphi(0)\cdot G(c) = (G(c)\tilde{\delta};\varphi(x))$$

✓

$$\frac{dG}{dc} = \left|\begin{array}{l} G(1)=0 \;\;\&\;\; \text{integrand } \lambda\text{-měřitelný (spojitý)}\\ \left|\frac{\partial g}{\partial c}\right| = |e^{-y}2\cdot\cos(cy)\cdot\sin(cy)| = |e^{-y}\sin(2cy)| \leq e^{-y}\in\mathcal{L}(\mathbb{R})\end{array}\right| =$$

$$= -\int_0^\infty e^{-y}\sin(2cy)\,dy = \frac{-2c}{1+4c^2} \;✓\; \Rightarrow\; G(c)=\frac{1}{4}\ln(1+4c^2)+\beta \;✓$$

$$G(1)\overset{!}{=}0 \;\Rightarrow\; G(1)=-\frac{1}{4}\ln 5+\beta \overset{!}{=} 0 \;\Rightarrow\; \beta=\frac{1}{4}\ln 5$$

$$\Rightarrow\; G(c)=\frac{1}{4}\ln 5-\frac{1}{4}\ln(1+4c^2)=\frac{1}{4}\ln\frac{5}{1+4a^2} \;✓$$

$$\lim_{\omega\to+\infty}\tilde{f}_\omega = \frac{1}{4}\ln\frac{5}{1+4c^2}\cdot\tilde{\delta}$$

$$\varphi(x) = \mu \int_0^x y\,\varphi(y)\,dy + x$$

princip metody

$$\varphi_0(x) = x$$

$$\varphi_1(x) = \mu \int_0^x y \cdot y\,dy + x = \frac{1}{3}\mu x^3 + x$$

$$\varphi_2(x) = \mu^2 \int_0^x y\,\frac{1}{3}y^3\,dy + \frac{1}{3}\mu x^3 + x = \mu^2 \cdot \frac{1}{5}\cdot\frac{1}{3}x^5 + \frac{1}{3}\mu x^3 + x$$

$$\varphi_3(x) = \mu^3 \int_0^x y \cdot \frac{1}{5}\cdot\frac{1}{3}y^5\,dy + \frac{1}{5}\cdot\frac{1}{3}\cdot\mu^2 x^5 + \frac{1}{3}\mu x^3 + x =$$

$$= \frac{1}{7}\cdot\frac{1}{5}\cdot\frac{1}{3}\mu^3 x^7 + \frac{1}{5}\cdot\frac{1}{3}\cdot\mu^2 x^5 + \frac{1}{3}\mu x^3 + x$$

Podezření:

$$\varphi_n(x) = \sum_{k=0}^{n} \frac{\mu^k}{(2k+1)!!} x^{2k+1}$$

Důkaz:

$$\varphi_{n+1}(x) = \mu \int_0^x y\,\varphi_n(y)\,dy + x = \mu \int_0^x \sum_{k=0}^{n} \frac{\mu^k}{(2k+1)!!} y^{2k+2}\,dy + x =$$

$$= \sum_{k=0}^{n} \frac{\mu^{k+1}}{(2k+1)!!} \int_0^x y^{2k+2}\,dy + x = \sum_{k=0}^{n} \frac{\mu^{k+1}}{(2k+1)!!}\,\frac{x^{2k+3}}{2k+3} + x =$$

$$= \sum_{k=0}^{n} \frac{\mu^{k+1}}{(2k+3)!!} x^{2k+3} + x = \left| \begin{matrix} l = k+1 \\ k = l-1 \end{matrix} \right| = \sum_{l=1}^{n+1} \frac{\mu^l}{(2l+1)!!} x^{2l+1} + x = \sum_{l=0}^{n+1} \frac{\mu^l}{(2l+1)!!} x^{2l+1}$$

q.e.d

Řešení:

$$\varphi(x) = \lim_{n\to\infty} \varphi_n(x) = \sum_{l=0}^{\infty} \frac{\mu^l}{(2l+1)!!} x^{2l+1} \quad \& \quad \varphi(0) = 0$$

$$\varphi'(x) = \sum_{l=0}^{\infty} \frac{\mu^l}{(2l-1)!!} x^{2l} = 1 + x \cdot \sum_{l=1}^{\infty} \frac{\mu^l}{(2l-1)!!} x^{2l-1} = \left| \begin{matrix} m = l-1 \\ l = m+1 \end{matrix} \right| =$$

$$= 1 + x \cdot \sum_{m=0}^{\infty} \frac{\mu^{m+1}}{(2m+1)!!} x^{2m+1} = 1 + x \cdot \mu \cdot \varphi(x)$$

ODR: $\varphi' - x \cdot \mu \cdot \varphi = 1 \quad / \, e^{-\mu \frac{x^2}{2}}$

$$\left(\varphi \cdot e^{-\mu \frac{x^2}{2}}\right)' = e^{-\mu \frac{x^2}{2}}$$

$$\Rightarrow \quad \varphi(x) = e^{\mu \frac{x^2}{2}} \cdot \int_0^x e^{-\mu \frac{t^2}{2}}\,dt = \omega_\mu(x) \cdot e^{\mu \frac{x^2}{2}}$$

Bianka Csanaková

(5) $(f_m, \varphi(x)) := \int_{-\infty}^{m} x^2 \varphi(x)\,dx$

$f_m \overset{?}{\in} \mathcal{D}'(\mathbb{R})$

↳ 1) existence $\int_{-\infty}^{m} x^2 \varphi(x)\,dx = \int_{-R}^{m} \underbrace{x^2 \cdot \varphi(x)}\,dx$ (supp $\varphi \subset \langle -R, R\rangle$)

↳ $\varphi(x) \in \mathcal{D}(\mathbb{R}) \Rightarrow$ je spojitá

$x^2 \in C^\infty(\mathbb{R})$

$x^2\varphi(x)$ je spojitá, $\langle -R, m\rangle$ je kompakt

$\Rightarrow$ integrál existuje

2) linearita → plyne z linearity integrálu

3) spojitost

předpokládáme, že $\varphi_k \xrightarrow{\mathcal{D}} 0$

↳ supp(φ_k) je omezený $\forall k \in \mathbb{N}$

$D^\alpha \varphi_k \overset{\mathbb{R}}{\rightrightarrows} 0$

$$\lim_{k\to\infty} \int_{-\infty}^{m} x^2 \varphi_k(x)\,dx = \int_{-\infty}^{m} x^2 \cdot 0\,dx = 0 \Rightarrow \lim_{k\to\infty} (f_m, \varphi_k(x)) = (f_m, 0) \text{ platí}$$

↳ $|x^2 \varphi_k(x)| \le \underbrace{K x^2}_{\text{integrabilní majoranta}} \in \mathcal{L}(S_R)$

$\Rightarrow$ z 1, 2, 3 plyne, že $f_m \in \mathcal{D}'(\mathbb{R})$

$$(f_m'', \varphi(x)) = (-1)^2 \cdot (f_m, \varphi''(x)) = \int_{-\infty}^{m} x^2 \varphi''(x)\,dx = \left| \begin{matrix} v' = \varphi'' & u = x^2 \\ v = \varphi' & u' = 2x \end{matrix} \right| =$$

$$= [x^2 \varphi']_{-\infty}^{m} - \int_{-\infty}^{m} 2x\varphi'\,dx = \left| \begin{matrix} v' = \varphi' & u = 2x \\ v = \varphi & u' = 2 \end{matrix} \right| =$$

$\lim_{x\to\infty} \varphi'(x) = 0$
$\varphi' \in \mathcal{D}(\mathbb{R})$ má omezený nosič

$\lim_{x\to\infty} \varphi(x) = 0$
$\varphi(x) \in \mathcal{D}(\mathbb{R}) \Rightarrow$ má omezený nosič, v ∞ je nulová

$$= m^2\varphi'(m) - [2x\varphi(x)]_{-\infty}^{m} + \int_{-\infty}^{m} 2\varphi(x)\,dx = m^2\varphi'(m) - 2m\varphi(m) + \int_{-\infty}^{m} 2\varphi(x)\,dx$$

$$= (-m^2\delta_m', \varphi) - (2m\delta_m, \varphi) + (\overbrace{\Theta_m}^{\text{nově označíme}}(-x) \cdot 2, \varphi(x)), \text{ kde } \overset{\text{ozn.}}{(\Theta_m(-x), \varphi)} = \int_{-\infty}^{m} \varphi(x)\,dx$$

$\Theta_m = \int_{m}^{+\infty} \varphi(x)\,dx$

$\Theta(-x+m) =$

$$= (\Theta_m(-x) - m^2\delta_m' - 2m\delta_m, \varphi(x))$$

platí $\forall \varphi(x) \in \mathcal{D}(\mathbb{R})$

$$\Rightarrow f_m'' = 2\Theta_m(-x) - m^2\delta_m' - 2m\delta_m, \text{ kde ozn. } (\Theta_m(-x), \varphi(x)) = \int_{-\infty}^{m} \varphi(x)\,dx$$

$$4x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 4xy \frac{\partial^2 u}{\partial x \partial y} - y \frac{\partial u}{\partial y} - 8u = 0$$ 8b

$d(x,y) = 16x^2y^2 - 16x^2y^2 = 0 \Rightarrow$ rovnice je všude parabolická!

$\lambda(x,y) = -\frac{4xy}{8x^2} = -\frac{y}{2x} \qquad y' = \frac{y}{2x} \Rightarrow \ln|y| = \ln C\sqrt{|x|} \Rightarrow \tilde{C} = \frac{y^2}{x}$

substituce: $\xi = y^2/x$, $\eta = x$

regularita: $\det\left(\frac{D(\xi,\eta)}{D(x,y)}\right) = \begin{vmatrix} -\frac{y^2}{x^2} & \frac{2y}{x} \\ 1 & 0 \end{vmatrix} = -2\frac{y}{x} \neq 0$

$H_{reg} = \{(x,y) \in \mathbb{R}^2 : y \neq 0 \wedge x \neq 0\}$

$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial \xi}\frac{y^2}{x^2} + \frac{\partial u}{\partial \eta} \quad \wedge \quad \frac{\partial u}{\partial y} = \frac{2y}{x}\frac{\partial u}{\partial \xi} \qquad \Big| (-y)$

$\frac{\partial^2 u}{\partial x^2} = \frac{y^4}{x^4}\frac{\partial^2 u}{\partial \xi^2} - 2\frac{y^2}{x^2}\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} + 2\frac{y^2}{x^3}\frac{\partial u}{\partial \xi} \qquad \Big| 4x^2$

$\frac{\partial^2 u}{\partial y^2} = 4\frac{y^2}{x^2}\frac{\partial^2 u}{\partial \xi^2} + \frac{2}{x}\frac{\partial u}{\partial \xi} \qquad \Big| y^2$

$\frac{\partial^2 u}{\partial x \partial y} = -2\frac{y^3}{x^3}\frac{\partial^2 u}{\partial \xi^2} + 2\frac{y}{x}\frac{\partial^2 u}{\partial \xi \partial \eta} - 2\frac{y}{x^2}\frac{\partial u}{\partial \xi} \qquad \Big| 4xy$

Dosazení:

$$\frac{\partial^2 u}{\partial \xi^2}\left(4\frac{y^4}{x^2} + 4\frac{y^4}{x^2} - 8\frac{y^4}{x^2}\right) + \frac{\partial^2 u}{\partial \xi \partial \eta}\left(-8y^2 + 8y^2\right) + \frac{\partial^2 u}{\partial \eta^2}\left(4x^2\right) +$$
$$+ \frac{\partial u}{\partial \xi}\left(8\frac{y^2}{x} - 8\frac{y^2}{x}\right) - \frac{2y^2}{x}\frac{\partial u}{\partial \xi} - 8u + 2\frac{y^2}{x}\frac{\partial u}{\partial \xi} = 0$$

$$\eta^2 \frac{\partial^2 u}{\partial \eta^2} - 2u(\xi,\eta) = 0$$

$\eta = e^t \overset{\eta > 0}{\Rightarrow} t = \ln \eta \Rightarrow \frac{du}{d\eta} = \frac{du}{dt}\cdot\frac{1}{\eta} \Rightarrow \frac{d^2u}{d\eta^2} = -\frac{1}{\eta^2}\dot{u} + \frac{1}{\eta^2}\ddot{u}$

$\ddot{u} - \dot{u} - 2u = 0 \qquad \lambda^2 - \lambda - 2 = 0 \qquad \lambda_1 = 2;\ \lambda_2 = -1$

$F_S = \{e^{2t};\ e^{-t}\} \Rightarrow u(\xi,\eta) = C(\xi)\cdot\eta^2 + D(\xi)\cdot\frac{1}{\eta}$

$\eta = -e^t \overset{\eta < 0}{\Rightarrow} t = \ln \eta \Rightarrow$ stejné!

$$u(\xi,\eta) = C(\xi)\cdot\eta^2 + D(\xi)\cdot\frac{1}{\eta}$$
$$u(x,y) = \frac{1}{x}D\left(\frac{y^2}{x}\right) + x^2\cdot C\left(\frac{y^2}{x}\right) \text{ na } H_{reg}$$