Katedra matematiky Fakulty jaderné a fyzikálně inženýrské ČVUT v Praze							CELKEM
Příjmení a jméno	1	2	3	4	5	6	

Zápočtová písemná práce č. 1 z předmětu 01RMF – varianta A

čtvrtek 19. listopadu 2015, 13:20–15:20

1 (5 bodů)
Nechť $f(x), g(x) \in \mathscr{L}_1(\mathbf{R})$ a

$$\int_{\mathbf{R}} f(x)\,\mathrm{d}x = A, \quad \int_{\mathbf{R}} xf(x)\,\mathrm{d}x = \mu, \quad \int_{\mathbf{R}} g(x)\,\mathrm{d}x = B, \quad \int_{\mathbf{R}} xg(x)\,\mathrm{d}x = \nu.$$

Vypočítejte, čemu se rovná $\int_{\mathbf{R}} z(f \star g)(z)\,\mathrm{d}z$.

2 (10 bodů)
Metodou iterovaných jader (tedy aplikací rezolventy) řešte integrální rovnici

$$\varphi(x) = \mu \int_0^x \sqrt{xy}\,\varphi(y)\,\mathrm{d}y + \sqrt{x}.$$

Tvar iterovaného jádra prokažte indukcí. Odlišná metoda řešení je nepřípustná!

3 (4 body)
Nechť posloupnost $(\varphi_k(\vec{x}))_{k=1}^{\infty}$ konverguje podle normy v Banachově prostoru $\mathscr{C}_\sigma(J)$ k nulové funkci. Dokažte, že pak

$$\varphi_k(\vec{x}) \overset{J}{\rightrightarrows} 0.$$

V důkaze komentujte, proč by tvrzení neplatilo, kdyby J nebyla kompaktní množina.

4 (11 bodů)
Řešte parciální diferenciální rovnici

$$3x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} - y^2\frac{\partial^2 f}{\partial y^2} - 4y\frac{\partial f}{\partial y} = 0.$$

5 (4 body)
Vypočítejte konvoluci $\Theta(x)x^m \star \mathrm{e}^x$, kde $m \in \mathbf{N}$.

6 (6 bodů)
Na posloupnosti se členy

$$g_n(x) = \left(\frac{x+2}{7}\right)^n$$

demonstrujte, že operátor $\hat{L} = \frac{\mathrm{d}\bullet}{\mathrm{d}x}$ není omezený ani spojitý v $\mathbb{L}_2(-2,5)$.

$\int_R f(x)\,dx = A$; $\int_R x f(x)\,dx = \mu$; $\int_R g(x)\,dx = B$; $\int_R x\, g(x)\,dx = \nu$

$$\int_R z\,(f*g)(z)\,dz = \int_R z \int_R f(x)\,g(z-x)\,dx\,dz = \int_{R^2} z\cdot f(x)\cdot g(z-x)\,d(x,z) =$$

$$= \left| \begin{matrix} z-x=y \\ dz=dy \end{matrix} \right| = \int_{R^2} f(x)\,(x+y)\cdot g(y)\,d(x,y) =$$

$$= \int_{R^2} x f(x)\, g(y)\,dy\,dx + \int_{R^2} y\, g(y) f(x)\,dx\,dy = \text{| FUBINIOVA VĚTA |}$$

$$= B\int_R x f(x)\,dx + A\int_R y\, g(y)\,dy = \underline{\mu B + \nu A}$$

$$\varphi(x) = \mu \int_0^x \sqrt{xy}\, \varphi(y)\, dy + \sqrt{x}$$

$$K_1(x,y) = \sqrt{xy}$$

$$K_2(x,y) = \int_y^x K_1(x,z) K_1(z,y)\, dz = \int_y^x \sqrt{x\cdot z}\cdot\sqrt{z\cdot y}\, dz = \sqrt{xy}\left[\frac{z^2}{2}\right]_y^x =$$

$$= \frac{1}{2}\sqrt{xy}\,(x^2-y^2)$$

$$K_3(x,y) = \int_y^x \sqrt{x\cdot z}\ \frac{1}{2}\sqrt{z\cdot y}\,(z^2-y^2)\, dz = \frac{1}{2}\sqrt{xy}\int_y^x z(z^2-y^2)\, dz$$

$$= \frac{1}{2}\sqrt{xy}\left[\frac{(z^2-y^2)^2}{4}\right]_y^x = \frac{1}{8}\sqrt{xy}\,(x^2-y^2)^2$$

<u>Hypotéza:</u>

$$\boxed{K_n(x,y) = \frac{1}{2^{n-1}}\frac{1}{(n-1)!}\sqrt{xy}\,(x^2-y^2)^{n-1}}$$

<u>Důkaz:</u>

$$K_{n+1}(x,y) = \int_y^x \sqrt{x\cdot z}\cdot\frac{1}{2^{n-1}}\frac{1}{(n-1)!}\sqrt{z\cdot y}\,(z^2-y^2)^{n-1}\, dz =$$

$$= \frac{1}{2^{n-1}}\frac{1}{(n-1)!}\sqrt{xy}\int_y^x z(z^2-y^2)^{n-1}\, dz = \frac{\sqrt{xy}}{2^{n-1}(n-1)!}\left[\frac{(z^2-y^2)^n}{n}\right]_y^x =$$

$$= \frac{\sqrt{xy}}{2^n\cdot n!}(x^2-y^2)^n \qquad \text{q.e.d.}$$

<u>Resolventa:</u>

$$R(x,y/\mu) = \sum_{k=0}^{\infty}\mu^k K_{k+1}(x,y) = \sum_{k=0}^{\infty}\mu^k\frac{\sqrt{xy}}{2^k\cdot k!}(x^2-y^2)^k =$$

$$= \sqrt{xy}\, e^{\mu(x^2-y^2)/2}$$

<u>Výsledek:</u>

$$\varphi(x) = \sqrt{x} + \mu\int_0^x \sqrt{xy}\, e^{\mu(x^2-y^2)\cdot\frac{1}{2}}\sqrt{y}\, dy = \sqrt{x} + \mu\sqrt{x}\int_0^x y\, e^{\mu(x^2-y^2)\cdot\frac{1}{2}}\, dy =$$

$$= \left|\begin{array}{l} u = \mu(x^2-y^2)\cdot\frac{1}{2} \\ du = -\mu y\, dy \end{array}\right| = \sqrt{x} + \sqrt{x}\int_0^{\mu x^2/2} e^u\, du = \sqrt{x} + \sqrt{x}\left[e^{\mu\frac{x^2}{2}} - 1\right] =$$

$$= \sqrt{x}\, e^{\mu x^2/2}$$

TU Delft

$$\lim_{n\to\infty} \text{norm}\, \varphi_n(\vec{x}) \overset{C_0(J)}{=} 0 \quad \overset{?}{\Rightarrow} \quad \varphi_n(\vec{x}) \overset{J}{\rightrightarrows} 0$$

$\downarrow$

$$\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}: \quad n > n_0 \Rightarrow \|\varphi_n(\vec{x}) - 0\| = \max_{\vec{x}\in J} |\varphi_n(\vec{x})| < \varepsilon$$

Chceme ukázat, že $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}: \ \vec{x} \in J \wedge n > n_0 \Rightarrow |\varphi_n(\vec{x}) - 0| < \varepsilon$

- pro pevně zvolené $\varepsilon > 0$ je maximální hodnota funkce $|\varphi_n(\vec{x})|$ jistě menší než ε

pro $n > n_0$ (of course)

$$\Rightarrow (\forall \vec{x} \in J)(\forall n > n_0): \ |\varphi_n(\vec{x})| < \varepsilon$$

- kdyby J nebyla kompaktní množina, pak by $\max_{\vec{x}\in J} |\varphi_n(\vec{x})|$ nebyla normou

$$3x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} - y^2\frac{\partial^2 f}{\partial y^2} - 4y\frac{\partial f}{\partial y} = 0$$

$d(x,y) = 4x^2y^2 + 12x^2y^2 = 16x^2y^2 > 0$ ... hyperbolická s.v.

$$\lambda(x,y) = \frac{-2xy \pm 4xy}{6x^2} = \left\langle \begin{array}{l} -y/x \\ \frac{1}{3}\, y/x \end{array}\right. \qquad \frac{dy}{dx} = -\frac{y}{x} \,;\, \frac{1}{3}\frac{y}{x}$$

$$\xi = \frac{y}{x}$$
$$\eta = xy^3$$

$$\det\left(\frac{D(\xi,\eta)}{D(x,y)}\right) = \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ y^3 & 3xy^2 \end{vmatrix} = -\frac{4y^3}{x}$$

$M_{reg} = \{(x,y)\in\mathbb{R}^2 : x\neq 0 \wedge y\neq 0\}$

(je-li jen determinant, tak půl bodu)

$$\frac{\partial f}{\partial x} = -\frac{y}{x^2}\frac{\partial f}{\partial \xi} + y^3\frac{\partial f}{\partial \eta}$$

$$-4y \;\Big/\; \frac{\partial f}{\partial y} = \frac{1}{x}\frac{\partial f}{\partial \xi} + 3xy^2\frac{\partial f}{\partial \eta}$$

$$3x^2 \;\Big/\; \frac{\partial^2 f}{\partial x^2} = \frac{y^2}{x^4}\frac{\partial^2 f}{\partial \xi^2} + y^6\frac{\partial^2 f}{\partial \eta^2} - 2\frac{y^4}{x^2}\frac{\partial^2 f}{\partial\xi\partial\eta} + 2\frac{y}{x^3}\frac{\partial f}{\partial \xi}$$

$$-y^2 \;\Big/\; \frac{\partial^2 f}{\partial y^2} = \frac{1}{x^2}\frac{\partial^2 f}{\partial \xi^2} + 9x^2y^4\frac{\partial^2 f}{\partial \eta^2} + 6y^2\frac{\partial^2 f}{\partial\xi\partial\eta} + 6xy\frac{\partial f}{\partial \eta}$$

$$2xy \;\Big/\; \frac{\partial^2 f}{\partial x\partial y} = -\frac{y}{x^3}\frac{\partial^2 f}{\partial \xi^2} + 3xy^5\frac{\partial^2 f}{\partial \eta^2} - 2\frac{y^3}{x}\frac{\partial^2 f}{\partial\xi\partial\eta} - \frac{1}{x^2}\frac{\partial f}{\partial \xi} + 3y^2\frac{\partial f}{\partial \eta}$$

$$y^4\frac{\partial^2 f}{\partial\xi\partial\eta}\left(-6-6-4\right) + \frac{\partial f}{\partial \xi}\left(-4\frac{y}{x} + 6\frac{y}{x} - 2\frac{y}{x}\right) + \frac{\partial f}{\partial \eta}\left(-12xy^3 - 6xy^3 + 6xy^3\right) = 0$$

$$-16y^4\frac{\partial^2 f}{\partial\xi\partial\eta} - 12xy^3\frac{\partial f}{\partial \eta} = 0$$

$$\frac{\partial^2 f}{\partial\xi\partial\eta} + \frac{3}{4}\frac{x}{y}\frac{\partial f}{\partial \eta} = 0$$

$$\frac{\partial^2 f}{\partial\xi\partial\eta} + \frac{3}{4}\frac{1}{\xi}\frac{\partial f}{\partial \eta} = 0 \qquad u = \frac{\partial f}{\partial \eta}$$

$$\frac{du}{d\xi} + \frac{3}{4\xi}u = 0$$

$$\frac{du}{d\xi} = -\frac{3}{4\xi}u$$

$$\frac{1}{u}\,du = -\frac{3}{4}\frac{1}{\xi}\,d\xi$$

$$\frac{4}{3}\ln|u| = -\ln|\xi| + C$$

$$u = \frac{1}{\xi^{3/4}}\cdot C(\eta)$$

$$f(\xi,\eta) = \frac{1}{\xi^{3/4}}C(\eta) + D(\xi)$$

$$f(x,y) = \left(\frac{x}{y}\right)^{3/4} C(xy^3) + D\left(\frac{y}{x}\right)$$

$$\theta(x)\cdot x^m * e^x = \int_{\mathbb{R}} \theta(s)\cdot s^m \cdot e^{x-s}\,ds = e^x \int_0^\infty s^m \cdot e^{-s}\,ds = m!\,e^x$$

$$\int_0^\infty e^{-\alpha x}\,dx = \frac{1}{\alpha} \quad \Big/ \frac{d^m}{d\alpha^m}$$

$$\underbrace{(-1)^m \cdot \int_0^\infty x^m e^{-\alpha x}\,dx = (-1)^m \frac{m!}{\alpha^{m+1}}}_{\int_0^\infty x^m e^{-x}\,dx = m!}$$

$$g_n(x) \to 0 \quad \Leftrightarrow \quad \forall \varepsilon > 0\ \exists n_0 : \ n > n_0 \Rightarrow \| g_n(x) - 0 \| < \varepsilon$$

zdůvodnění:
$$\| g_n(x) - 0 \|^2 = \langle g_n | g_n \rangle = \int_{-2}^{5} | g_n(x) |^2 dx =$$

$$= \int_{-2}^{5} \left(\frac{x+2}{7} \right)^{2n} dx = \frac{1}{49^n} \left[\frac{(x+2)^{2n+1}}{2n+1} \right]_{-2}^{5} = \frac{7^{2n+1}}{7^{2n}} \cdot \frac{1}{2n+1} = \frac{7}{2n+1}$$

$$\lim_{n \to \infty} \frac{7}{2n+1} = 0 \Rightarrow$$ definice konvergence podle normy je naplněna

$$\hat{L}(g_n) = g_n'(x) \not\to 0 \qquad g_n'(x) = n \left(\frac{x+2}{7} \right)^{n-1} \cdot \frac{1}{7}$$

$$\| g_n'(x) \|^2 = \langle \hat{L} g_n | \hat{L} g_n \rangle = \int_{-2}^{5} \frac{n^2}{49} \left(\frac{x+2}{7} \right)^{2n-2} dx =$$

$$= \frac{n^2}{7^{2n}} \left[\frac{(x+2)^{2n-1}}{2n-1} \right]_{-2}^{5} = \frac{7^{2n-1}}{7^{2n}} \frac{n^2}{2n-1} = \frac{n^2}{14n-7}$$

$$\lim_{n \to \infty} \frac{n}{14n-7} = \frac{1}{14} \neq 0 \Rightarrow \hat{L}(g_n) \not\to 0$$

$\Rightarrow$ $\hat{L}$ není spojitý, a tedy ani omezený

Příjmení a jméno	1	2	3	4	5	6	CELKEM

Zápočtová písemná práce č. 1 z předmětu 01RMF – varianta B

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1 (10 bodů)
Řešte parciální diferenciální rovnici

$$x\frac{\partial^2 u}{\partial x^2} - x^3y^2\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} - x^3y\frac{\partial u}{\partial y} = 4x^3y^4.$$

2 (4 body)
Na posloupnosti se členy

$$g_n(x) = \mathrm{e}^{-nx^2}$$

demonstrujte, že i posloupnost, která stejnoměrně nekonverguje na $\mathbf{R}$ může konvergovat podle normy na $\mathscr{L}_2(\mathbf{R})$.

3 (5 bodů)
Dokažte, že konverguje-li posloupnost $(\varphi_k(\vec{x}))_{k=1}^{\infty}$ na G stejnoměrně k nulové funkci, kde $\mu(G) < \infty$, pak také konverguje podle normy v $\mathscr{L}_2^{(w)}(G)$. V důkaze vysvětlete, jaký zádadní předpoklad o váze $w(\vec{x})$ je třeba učinit a proč.

4 (5 bodů)
Vypočítejte konvoluci $x^2 \star \mathrm{e}^{-4x^2}$.

5 (10 bodů)
Metodou iterovaných jader (tedy aplikací rezolventy) řešte integrální rovnici

$$\varphi(x) = \mu\int_0^x \sqrt{xy}\,\varphi(y)\,\mathrm{d}y + \sqrt{x}.$$

Tvar iterovaného jádra prokažte indukcí. Odlišná metoda řešení je nepřípustná!

6 (6 bodů)
Nechť $\hat{S} : \mathscr{H} \mapsto \mathscr{H}$ je operátor s čistě bodovým spekrem, $(\lambda_1, \lambda_2, \lambda_3, \ldots)$ jeho unfoldované spektrum a $B = \{\varphi_1(x), \varphi_2(x), \varphi_3(x), \ldots\}$ příslušná operátorová báze. Odvoďte univerzální formule pro hodnoty

- a) $\langle \hat{S}(f) | \hat{S}(g) \rangle$,
- b) $||\hat{S}(f)||$

platné pro libovolné funkce $f(x), g(x) \in \mathscr{H}$. V jejich tvaru užijte Fourierovy koeficienty. Ve výpočtu komentujte každou vlastnost, jíž využíváte!

$$\frac{\partial^2 u}{\partial x^2} - x^2y^2\frac{\partial^2 u}{\partial y^2} - \frac{1}{x}\frac{\partial u}{\partial x} \neq 4x^2y^4 \qquad -x^2y\frac{\partial u}{\partial y}$$

$$d(x,y) = 4x^2y^2 = \begin{cases} 0 \text{ na } G_P = \{(x,y)\in\mathbb{R}^2 : x=0 \vee y=0\} \\ >0 \text{ na } G_H = \{(x,y)\in\mathbb{R}^2 : x\neq 0 \wedge y\neq 0\} \end{cases} \qquad G_E = \emptyset$$

$$\lambda(x,y) = \pm 2xy\cdot\frac{1}{2} \Rightarrow y' = \pm xy \Rightarrow y' \pm xy = 0 \qquad e^{\pm\frac{x^2}{2}}$$

$$\left(y\cdot e^{\pm\frac{x^2}{2}}\right)' = c' \Rightarrow \boxed{\begin{aligned} \xi &= y\cdot e^{x^2/2} \\ \eta &= y\cdot e^{-x^2/2}\end{aligned}}$$

$$\det\left(\frac{D(\xi,\eta)}{D(x,y)}\right) = \begin{vmatrix} xye^{x^2/2} & e^{x^2/2} \\ -xye^{-x^2/2} & e^{-x^2/2}\end{vmatrix} = xy + xy = 2xy \neq 0 \text{ na } G_H$$

$$\Rightarrow M_{reg} = G_H$$

$$\frac{\partial u}{\partial x} = xye^{x^2/2}\frac{\partial u}{\partial \xi} - xye^{-x^2/2}\frac{\partial u}{\partial \eta} \qquad \frac{\partial u}{\partial y} = e^{x^2/2}\frac{\partial u}{\partial \xi} + e^{-x^2/2}\frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2} = x^2y^2e^{x^2}\frac{\partial^2 u}{\partial \xi^2} + x^2y^2e^{-x^2}\frac{\partial^2 u}{\partial \eta^2} - 2x^2y^2\frac{\partial^2 u}{\partial \xi\partial \eta} + ye^{x^2/2}\frac{\partial u}{\partial \xi} - ye^{-x^2/2}\frac{\partial u}{\partial \eta} + x^2ye^{x^2/2}\frac{\partial u}{\partial \xi} + x^2ye^{-x^2/2}\frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial y^2} = e^{x^2}\frac{\partial^2 u}{\partial \xi^2} + e^{-x^2}\frac{\partial^2 u}{\partial \eta^2} + 2\frac{\partial^2 u}{\partial \xi\partial \eta}$$

Dakle: $\quad -4x^2y^2\dfrac{\partial^2 u}{\partial \xi\partial \eta} = 4x^2y^4$

$$\boxed{\frac{\partial^2 u}{\partial \xi\partial \eta} = -\xi\cdot\eta}$$

$$\frac{\partial u}{\partial \xi} = -\frac{1}{2}\xi\eta^2 + C(\xi) \Rightarrow u(\xi,\eta) = -\frac{1}{4}\xi^2\eta^2 + C(\xi) + D(\eta)$$

$$\boxed{u(x,y) = -\frac{1}{4}y^4 + C\left(y\cdot e^{x^2/2}\right) + D\left(y\cdot e^{-x^2/2}\right)}$$

$$e^{-nx^2} \longrightarrow \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases} \qquad \text{(není spojitá)}$$

$$\sigma_n = \sup_{x \in \mathbb{R}} |g_n(x) - g(x)| = \sup_{x \neq 0} |e^{-nx^2}| = 1 \neq 0$$

$$\Rightarrow \text{ podle supremálního kritéria: } \quad g_n(x) \overset{\mathbb{R}}{\not\rightrightarrows} g(x)$$

Ale:

$$\| e^{-nx^2} - 0 \|^2 = \langle e^{-nx^2} | e^{-nx^2} \rangle = \int_{\mathbb{R}} e^{-2nx^2} dx = \sqrt{\frac{\pi}{2n}} \rightarrow 0$$

– tedy jistě $\forall \varepsilon > 0 \; \exists n_0 \in \mathbb{N}: \; n > n_0 \Rightarrow \| e^{-nx^2} - 0 \| < \varepsilon$

$\varphi_k(\vec{x}) \overset{G}{\rightrightarrows} 0 \quad \Leftrightarrow \quad \forall \varepsilon > 0 \ \exists n_0 \in N: \ n > n_0 \wedge \vec{x} \in G \Rightarrow |\varphi_k(\vec{x})| < \frac{\varepsilon}{\sqrt{K \cdot \mu(G)}}$

Chci ukázat: $\quad \forall \varepsilon > 0 \ \exists n_0 \in N: \ n > n_0 \Rightarrow \|\varphi_n(\vec{x}) - 0\| < \varepsilon$

Tedy: $\|\varphi_n(\vec{x}) - 0\|^2 = \langle \varphi_n | \varphi_n \rangle = \int_G |\varphi_k(\vec{x})|^2 w(\vec{x})\, d\vec{x} \leq$

$$\leq \mu(G) \cdot \underbrace{\max_{\vec{x} \in \overline{G}} |w(\vec{x})|} \cdot \left(\max_{\vec{x} \in G} |\varphi_k(\vec{x})| \right)^2 < \mu(G) \cdot K \cdot \frac{\varepsilon^2}{K \cdot \mu(G)} = \varepsilon^2$$

váha $w(\vec{x})$ musí být buď omezená na G nebo spojitá na $\overline{G}$

$\Rightarrow \exists K \in R: \quad \forall \vec{x} \in G: \ |w(\vec{x})| = w(\vec{x}) < K$

nebo třídy $L^1(G)$, kde K značí $\int_G w(\vec{x})\, d\vec{x}$.

a zároveň $w(x) > 0$ s.v. na G,

aby $\| \cdot \|_w$ byla norma.

$$x^2 * e^{-4x^2} = \int_R (x-s)^2 e^{-4s^2} ds = x^2 \int_R e^{-4s^2} ds - 2x \int_R s \cdot e^{-4s^2} ds +$$

$$+ \int_R s^2 e^{-4s^2} ds = \alpha + \beta + \gamma$$

$$\alpha) \quad x^2 \int_R e^{-4s^2} ds = \left| \int_{-\infty}^{\infty} e^{-at^2} dt = \sqrt{\frac{\pi}{a}} \right| = x^2 \frac{\sqrt{\pi}}{2} = \frac{x^2}{2}\sqrt{\pi}$$

$$\beta) \quad -2x \int_R s \cdot e^{-4s^2} ds = 0 \quad \Leftarrow \text{lichý integrand}$$

$$\gamma) \quad \int_R s^2 e^{-4s^2} ds = \left| \begin{matrix} \int_R e^{-at^2} dt = \sqrt{\frac{\pi}{a}} \quad \Big/ \frac{d}{da} \\ \int_R t^2 e^{-at^2} dt = \frac{1}{2}\sqrt{\frac{\pi}{a^3}} \end{matrix} \right| = \frac{1}{2}\sqrt{\frac{\pi}{4^3}} = \frac{1}{16}\sqrt{\pi}$$

$$\alpha + \beta + \gamma = \frac{x^2}{2}\sqrt{\pi} + \frac{1}{16}\sqrt{\pi} = \frac{\sqrt{\pi}}{16}(1 + 8x^2)$$

$$\varphi(x) = \mu \int_0^x \sqrt{xy}\,\varphi(y)\,dy + \sqrt{x}$$

$$K_1(x,y) = \sqrt{xy}$$

$$K_2(x,y) = \int_y^x K_1(x,z)K_1(z,y)\,dz = \int_y^x \sqrt{x\cdot z}\cdot\sqrt{z\cdot y}\,dz = \sqrt{xy}\left[\frac{z^2}{2}\right]_y^x =$$

$$= \frac{1}{2}\sqrt{xy}\,(x^2-y^2)$$

$$K_3(x,y) = \int_y^x \sqrt{x\cdot z}\,\frac{1}{2}\sqrt{z\cdot y}\,(z^2-y^2)\,dz = \frac{1}{2}\sqrt{xy}\int_y^x z(z^2-y^2)\,dz$$

$$= \frac{1}{2}\sqrt{xy}\left[\frac{(z^2-y^2)^2}{4}\right]_y^x = \frac{1}{8}\sqrt{xy}\,(x^2-y^2)^2$$

Hypotéza:

$$\boxed{K_n(x,y) = \frac{1}{2^{n-1}}\frac{1}{(n-1)!}\sqrt{xy}\,(x^2-y^2)^{n-1}}$$

Důkaz:

$$K_{n+1}(x,y) = \int_y^x \sqrt{x\cdot z}\cdot\frac{1}{2^{n-1}}\frac{1}{(n-1)!}\sqrt{z\cdot y}\,(z^2-y^2)^{n-1}\,dz =$$

$$= \frac{1}{2^{n-1}}\frac{1}{(n-1)!}\sqrt{xy}\int_y^x z(z^2-y^2)^{n-1}\,dz = \frac{\sqrt{xy}}{2^{n-1}(n-1)!}\left[\frac{(z^2-y^2)^n}{n}\right]_y^x =$$

$$= \frac{\sqrt{xy}}{2^n n!}(x^2-y^2)^n \qquad \text{q.e.d.}$$

Rezolventa:

$$R(x,y|\mu) = \sum_{k=0}^{\infty}\mu^k K_{k+1}(x,y) = \sum_{k=0}^{\infty}\mu^k\frac{\sqrt{xy}}{2^k k!}(x^2-y^2)^k =$$

$$= \cancel{\frac{1}{4}}\sqrt{xy}\,e^{\mu(x^2-y^2)/2}$$

Výsledek:

$$\varphi(x) = \sqrt{x} + \mu\int_0^x \cancel{\frac{1}{4}}\sqrt{xy}\,e^{\mu(x^2-y^2)\cdot\frac{1}{2}}\sqrt{y}\,dy = \sqrt{x} + \mu\frac{\sqrt{xy}}{\cancel{2}}\int_0^x y\,e^{\mu(x^2-y^2)\cdot\frac{1}{2}}\,dy =$$

$$= \left|\begin{matrix} u = \mu(x^2-y^2)\cdot\frac{1}{2} \\ du = -\mu y\,dy \end{matrix}\right| = \sqrt{x} + \cancel{\frac{1}{4}}\sqrt{x}\int_0^{\mu\frac{x^2}{2}} e^u\,du = \sqrt{x} + \cancel{\frac{1}{4}}\sqrt{x}\left[e^{\mu\frac{x^2}{2}} - 1\right] =$$

$$= \sqrt{x}\,e^{\mu x^2/2}$$

a) $$\langle \hat{L}f | \hat{L}g \rangle \overset{1}{=} \left\langle \hat{L}\Big(\sum_{k=1}^{\infty} a_k \varphi_k(x)\Big) \,\Big|\, \hat{L}\Big(\sum_{l=1}^{\infty} b_l \varphi_l(\vec{x})\Big) \right\rangle \overset{2}{=}$$

$$\overset{2}{=} \left\langle \sum_{k=1}^{\infty} a_k \hat{L}(\varphi_k) \,\Big|\, \sum_{l=1}^{\infty} b_l \hat{L}(\varphi_l) \right\rangle \overset{3}{=} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \langle a_k \hat{L}(\varphi_k) | b_l \hat{L}(\varphi_l) \rangle \overset{4}{=}$$

$$\overset{4}{=} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \langle a_k \lambda_k \varphi_k(x) | b_l \lambda_l \varphi_l(\vec{x}) \rangle \overset{5}{=} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_k \lambda_k b_l^* \lambda_l^* \langle \varphi_k | \varphi_l \rangle \overset{6}{=}$$

$$\overset{6}{=} \sum_{k=1}^{\infty} a_k \lambda_k \lambda_k^* b_k^* \overset{7}{=} \underline{\sum_{k=1}^{\infty} a_k b_k^* |\lambda_k|^2}$$

b) $$\langle \hat{L}f | \hat{L}f \rangle = |\hat{L}f|^2 = \underline{\sum_{k=1}^{\infty} |a_k|^2 \cdot |\lambda_k|^2}$$

1 Fourierův rozvoj
2 spojitost operátoru (plyne z omezenosti)
3 spojitost skal. součinu (univerzální vlastnost)
4 definice vlastních hodnot a funkcí
5 vlastnost skalárního součinu
6 vlastnost funkcí v operátorovém spektru
7 úpravy v $\mathbb{C}$